IN-34

154697

P.46

NASA Contractor Report 191414

ICASE Report No. 92–73

ICASE

A MULTIPLE SCALES APPROACH TO SOUND GENERATION BY VIBRATING BODIES

(NASA-CR-191414) A MULTIPLE SCALES APPROACH TO SOUND GENERATION BY VIBRATING BODIES Final Report (ICASE) 46 p

N93-23409

Unclas

G3/34 0154697

James F. Geer

Dennis S. Pope

NASA Contract Nos. NAS1–18605 and NAS1–19480

December 1992

Institute for Computer Applications in Science and Engineering

NASA Langley Research Center

Hampton, Virginia 23681–0001

Operated by the Universities Space Research Association

National Aeronautics and Space Administration

Langley Research Center

Hampton, Virginia 23665-5225

A Multiple Scales Approach to
Sound Generation by Vibrating Bodies

James F. Geer [1]
Department of Systems Science
Watson School of Engineering and Applied Science
State University of New York
Binghamton, New York 13902

Dennis S. Pope
Lockheed Engineering and Sciences Company
144 Research Drive
Hampton, Virginia 23666

Abstract

The problem of determining the acoustic field in an inviscid, isentropic fluid generated by a solid body whose surface executes prescribed vibrations is formulated and solved as a multiple scales perturbation problem, using the Mach number M based on the maximum surface velocity as the perturbation parameter. Following the idea of multiple scales, new "slow" spacial scales are introduced, which are defined as the usual physical spacial scale multiplied by powers of M. The governing nonlinear differential equations lead to a sequence of linear problems for the perturbation coefficient functions. However, it is shown that the higher order perturbation functions obtained in this manner will dominate the lower order solutions unless their dependence on the slow spacial scales is chosen in a certain manner. In particular, it is shown that the perturbation functions must satisfy an equation similar to Burgers' equation, with a slow spacial scale playing the role of the time-like variable. The method is illustrated by a simple one-dimensional example, as well as by three different cases of a vibrating sphere. The results are compared with solutions obtained by purely numerical methods and some insights provided by the perturbation approach are discussed.

[1]The first author was supported by NASA Research Grant NAG-1-1099. Research was also supported by the National Aeronautics and Space Administration under NASA Contract Nos. NAS1-18605 and NAS1-19480 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23681-0001.

1. Introduction

We wish to describe the acoustic field in an inviscid, isentropic fluid generated by a solid body whose surface executes prescribed vibrations. When the Mach number M, based on the maximum surface velocity is small, it is natural to consider a perturbation approach to the problem. Crow (1970) carefully analyzed a general class of problems of this type, including Lighthill's (1952, 1954) acoustical analogy approach, involving quadrupole sources, and Ribner's (1962) approach, based on monopole sources. He showed how this class of problems could be treated as a singular perturbation problem by the method of inner and outer asymptotic expansions. Whitham (1956, 1974) examined this problem in the high frequency limit by using a perturbation approach similar to Lighthill's (1949, 1961) method of strained coordinates. He essentially "strained" the time coordinate and obtained expressions for the acoustic field valid in the far field for high frequency excitations. The approach we shall use is not restricted by the high frequency assumption, although our results do reduce to Whitham's results (to leading order in M) in the far field and at high frequencies. In addition, the analysis we shall present, being based on multiple *spacial* scales, provides insight into the *effects* of the nonlinearity on the form of the solution, as well as insights into the interaction of the value of the Mach number and the angular dependence of the solution (see section 9).

In section 2 we formulate the mathematical problem we wish to solve and then outline the multiple scales approach in section 3. We apply our method to a simple one-dimensional example in section 4. In section 5 we formulate the problem for a vibrating sphere and then determine the leading terms in the multiple scales expansion in sections 6 and 7. In section 8 we apply our results to three different vibrating spheres and then discuss our results in section 9.

2. Problem Formulation

We wish to describe the acoustic field in an inviscid, isentropic fluid generated by a solid body whose surface executes prescribed vibrations. To begin, we let the origin of a Cartesian coordinate system (x_1, x_2, x_3) be fixed at some convenient location inside the body. Then, in terms of these coordinates, the equations of conservation of mass and momentum are

$$\partial\rho/\partial t + \vec{\nabla}\bullet(\rho\vec{u}) = 0, \tag{2.1}$$

$$\partial p/\partial x_i + \rho\left(\partial u_i/\partial t + (\vec{u}\bullet\vec{\nabla})u_i\right) = 0, \quad i=1,2,3 \ . \tag{2.2}$$

In (2.1) and (2.2) ρ, u_i, p, and t represent the fluid density, the fluid velocity component in the positive x_i direction, the fluid pressure, and the time coordinate, respectively. Also, $\vec{u} = u_1\vec{i}_1 + u_2\vec{i}_2 + u_3\vec{i}_3$ and $\vec{\nabla} = \vec{i}_1(\partial/\partial x_1) + \vec{i}_2(\partial/\partial x_2) + \vec{i}_3(\partial/\partial x_3)$, where $\vec{i}_j$ is a unit vector in the positive x_j direction. We also let $p = k\rho^\gamma$, where k and γ are known constants, and define

$$c^2(\rho) \equiv dp/d\rho = k\gamma(\rho)^{\gamma-1}. \tag{2.3}$$

We let ρ_0, U, and L be typical (constant) values for the density, velocity, and length scales, respectively, associated with the flow, and define nondimensional variables (denoted by a "^" above the quantity) by

$$\hat{\rho} = \rho/\rho_0, \quad \hat{\vec{u}} = \vec{u}/U, \quad \hat{p} = (p-p_0)/(\rho_0U^2), \tag{2.4}$$
$$\hat{x}_j = x_j/L \ (j=1,2,3), \quad \hat{t} = t/(L/c_0),$$

where $p_0 = k\rho_0^\gamma$ and $c_0 = c(\rho_0)$. We now rewrite equations (2.1) and (2.2) in terms of these variables and then omit the "^" above the various quantities to obtain the relations

$$\partial\rho/\partial t + M\vec{\nabla}\bullet(\rho\vec{u}) = 0, \tag{2.5}$$

$$(2.6)\qquad \partial\rho/\partial x_i + \rho^{2-\gamma}\left(M\partial u_i/\partial t + M^2(\vec{u}\bullet\vec{\nabla})u_i\right) = 0,\ i=1,2,3\ ,$$

where $M \equiv U/c_0$ is the Mach number of the flow.

We now seek approximate solutions to equations (2.5) and (2.6), subject to appropriate boundary and initial conditions, which will be formally valid for small values of M.

3. A multiple scales perturbation solution of the basic equations

Following the method of multiple scales (see e.g. Nayfeh(1973), Chapter 5), we introduce the spacial scales (variables) $\vec{x}^{(k)}$, $k = 0, 1, 2, \ldots$, related to $\vec{x}$ by

$$(3.1)\qquad \vec{x}^{(k)} = M^k\vec{x},\ k = 0,\ 1,\ 2,\ \ldots .$$

Thus $\vec{x}^{(0)} = \vec{x}$, $\vec{x}^{(1)} = M\vec{x}$, $\vec{x}^{(2)} = M^2\vec{x}$, etc. We then assume that $\vec{u}$ and ρ are functions of these new scales, as well as functions of t and M, i.e. $\rho = \rho(t,\vec{x}^{(0)},\vec{x}^{(1)},\vec{x}^{(2)},\ldots,M)$ and $\vec{u} = \vec{u}(t,\vec{x}^{(0)},\vec{x}^{(1)},\vec{x}^{(2)},\ldots,M)$.

The method of multiple scales now treats all of the variables $\{x_i^{(k)}\}$ as <u>independent</u> variables and seeks to determine ρ and $\vec{u}$ as functions of these variables. In particular, for "small" values of M, we look for solutions in the form

$$(3.2)\qquad \begin{aligned} \rho &= 1 + M\rho^{(1)} + M^2\rho^{(2)} + \ldots = \sum_{j=0}^{\infty} \rho^{(j)}M^j\ , \quad \text{with } \rho^{(0)} \equiv 1, \\ \vec{u} &= \vec{u}^{(0)} + M\vec{u}^{(1)} + M^2\vec{u}^{(2)} + \ldots = \sum_{j=0}^{\infty} \vec{u}^{(j)}M^j\ . \end{aligned}$$

Here each of the coefficient functions $\rho^{(j)}$ and $\vec{u}^{(j)}$ is independent of M, but, in general, will depend upon t and the spacial scales $x_i^{(k)}$, i.e. $\rho^{(j)} = \rho^{(j)}(t,\vec{x}^{(0)},\vec{x}^{(1)},\ldots)$ and $\vec{u}^{(j)} = \vec{u}^{(j)}(t,\vec{x}^{(0)},\vec{x}^{(1)},\ldots)$.

To determine these coefficient functions, we substitute the expansions (3.2) into equations (2.5)-(2.6) and use the relation

$$\vec{\nabla} = \vec{\nabla}^{(0)} + M\vec{\nabla}^{(1)} + M^2\nabla^{(2)} + \ldots , \tag{3.3}$$

where $\quad \vec{\nabla}^{(k)} \equiv \vec{i}_1(\partial/\partial x_1^{(k)}) + \vec{i}_2(\partial/\partial x_2^{(k)}) + \vec{i}_3(\partial/\partial x_3^{(k)})$.

We then collect coefficients of like powers of M on the left side of each equation and, hence, express the left side of each equation as a power series in M. We then equate to zero the coefficient of each power of M, since the right side of each equation is zero. In this way, we obtain the following system of equations satisfied by the coefficient functions $\rho^{(j)}$ and $\vec{u}^{(j)}$:

$$\partial\rho^{(1)}/\partial t + \vec{\nabla}^{(0)}\bullet\vec{u}^{(0)} = 0, \tag{3.4a}$$

$$\partial\rho^{(1)}/\partial x_i^{(0)} + \partial u_i^{(0)}/\partial t = 0, \quad i = 1,\ 2,\ 3; \tag{3.4b}$$

$$\partial\rho^{(k+1)}/\partial t + \vec{\nabla}^{(0)}\bullet\vec{u}^{(k)} = F^{(k)}, \tag{3.5a}$$

$$\partial\rho^{(k+1)}/\partial x_i^{(0)} + \partial u_i^{(k)}/\partial t = G_i^{(k)}, \quad i = 1,\ 2,\ 3,\ k\geq 1. \tag{3.5b}$$

(Equations (3.4) follow from the terms in equations (2.5)-(2.6) which are O(M), while equations (3.5) follow from the terms in these equations which are $O(M^{k+1})$.) Here the functions $F^{(k)}$ and $G_i^{(k)}$ depend only upon $\rho^{(j)}$ with $j<k+1$ and $\vec{u}^{(j)}$ with $j<k$. In particular, we find

$$F^{(1)} = -\left(\vec{\nabla}^{(0)}\bullet(\rho^{(1)}\vec{u}^{(0)}) + \vec{\nabla}^{(1)}\bullet\vec{u}^{(0)}\right), \tag{3.6a}$$

(3.6b) $$G_i^{(1)} = -\left((\vec{u}^{(0)}\cdot\vec{\nabla}^{(0)})u_i^{(0)} + (2-\gamma)\rho^{(1)}\partial u_i^{(0)}/\partial t + \partial\rho^{(1)}/\partial x_i^{(1)}\right).$$

To solve equations (3.4), we set

(3.7) $$\rho^{(1)} = -\partial\varphi/\partial t \quad\text{and}\quad \vec{u}^{(0)} = \vec{\nabla}^{(0)}\varphi \ .$$

Then equations (3.4b) are satisfied for any choice of φ, while equation (3.4a) yields the requirement that φ must satisfy the linear wave equation

(3.8) $$\partial^2\varphi/\partial t^2 - \Delta^{(0)}\varphi = 0,$$

where $\Delta^{(0)} \equiv (\partial/\partial x_1^{(0)})^2 + (\partial/\partial x_2^{(0)})^2 + (\partial/\partial x_3^{(0)})^2$ is the usual Laplacian operator in the variables $\{x_i^{(0)}\}$.

From the structure of equations (3.5), we see that these equations can (in principle) be solved recursively, starting with k=1. In particular, equations (3.5) are a system of *linear* equations for the unknowns $\rho^{(k+1)}$ and $\vec{u}^{(k)}$. Consequently, we can express the solution to these equations as the superposition of a particular solution and a homogeneous solution to these equations. The homogeneous solution has the same form as the solution to equations (3.4) (see equations (3.7) and (3.8)). The particular solution will, of course, depend on the index k, as well as on the solutions for the lower order perturbation coefficients. In general, these particular solutions have the property (as we shall demonstrate explicitly in the following sections) that they tend to *decay more slowly in magnitude as* $|\vec{x}|\to\infty$ *than* $\rho^{(1)}$ *and* $\vec{u}^{(0)}$. Consequently, if this decay were to be left unchecked, the perturbation expansions (3.2) would become *invalid* as $|\vec{x}|\to\infty$. However, as we shall show, it is possible to satisfy the requirement that $\rho^{(k+1)}$ and $\vec{u}^{(k)}$ should *decay at least as fast as* $\rho^{(1)}$ and $\vec{u}^{(0)}$ as $|\vec{x}|\to\infty$ by properly choosing the dependence of $\rho^{(k+1)}$ and $\vec{u}^{(k)}$ on the "slower" spacial scales $\vec{x}^{(1)}$, $\vec{x}^{(2)}$, etc. Thus, this method will yield a perturbation expansion (3.2) which will be uniformly valid as $|\vec{x}|\to\infty$ and, consequently, will correctly represent the far field behavior of the acoustical radiation.

In the following section we shall demonstrate some of these ideas with a simple, one-dimensional example, and then proceed to a class of three-dimensional problems in sections 5-8.

4. A Simple One-Dimensional Example

As a simple example to illustrate the general ideas of our approach, we consider first a flow which varies in only one spacial coordinate, such as one-dimensional flow in a semi-infinite tube. We let this one spacial coordinate be denoted by $x_1 = x$, so that equations (2.5)-(2.6) become

$$\partial\rho/\partial t + M(\partial/\partial x)(\rho u) = 0 \ , \tag{4.1}$$

$$\partial\rho/\partial x + \rho^{2-\gamma}\Big(M\partial u/\partial t + M^2 u(\partial u/\partial x)\Big) = 0 \ . \tag{4.2}$$

We shall assume that u is a specified function, say $\tilde{f}(t)$, at x=0 for all $t\geq 0$, and that ambient conditions prevail at t=0 for all $0<x<\infty$, i.e.

$$u\Big|_{x=0} = \tilde{f}(t) \text{ for } t\geq 0, \text{ and } u\Big|_{t=0} = 0,\ \rho\Big|_{t=0} = 1\ , \text{ for } x>0. \tag{4.3}$$

Following the method of multiple scales outlined above, we find it convenient to define the differential operators D_i by

$$D_i \equiv \partial/\partial x^{(i)},\ i = 0,\ 1,\ \ldots\ . \tag{4.4}$$

We then look for solutions for ρ and u in the form of (3.2), where, in particular, $\rho^{(1)}$ and $u^{(0)}$ satisfy equations (3.4). Then the general solution to equation (3.8) is given by $\varphi = F(t-x) + G(t+x)$, where F and G are arbitrary functions of their arguments and may also depend on the spacial scales $x^{(1)}$, $x^{(2)}$, ... Since we shall restrict our attention to solutions which propagate only in the positive x direction, we set $G \equiv 0$ and write

$$\rho^{(1)} = u^{(0)} = f(t-x,\ x^{(1)},\ x^{(2)},\ \ldots), \tag{4.5}$$

where the exact form of the function f will be determined from the boundary and initial conditions (4.3) of the problem.

Using the solutions (4.5), equations (3.5) with k=1 for $\rho^{(2)}$ and $u^{(1)}$ become

$$\begin{aligned} \partial\rho^{(2)}/\partial t + D_0 u^{(1)} &= -\left(D_1 f + D_0(f^2)\right) \\ \partial u^{(1)}/\partial t + D_0 \rho^{(2)} &= -\left(D_1 f + ((\gamma-1)/2) D_0(f^2)\right). \end{aligned} \tag{4.6}$$

The general solution to (4.6) for $\rho^{(2)}$ and $u^{(1)}$ can be written as

$$\begin{aligned} \rho^{(2)} &= -x\left(D_1 f + ((1+\gamma)/4) D_0(f^2)\right) + f_1(t-x,\ x^{(1)},\ x^{(2)},\ \ldots), \\ u^{(1)} &= -x\left(D_1 f + ((1+\gamma)/4) D_0(f^2)\right) \\ &\qquad + ((\gamma-3)/4)(f^2) + f_1(t-x,\ x^{(1)},\ x^{(2)},\ \ldots). \end{aligned} \tag{4.7}$$

In (4.7), f_1 is an arbitrary function of its arguments, which will eventually be determined by the initial and boundary conditions of the problem.

In order for $\rho^{(2)}$ and $u^{(1)}$ to grow no faster that $\rho^{(1)}$ and $u^{(0)}$, respectively, as x becomes large, we must require that the term in brackets in equations (4.7) vanishes, i.e.

$$D_1 f + ((1+\gamma)/4) D_0(f^2) = 0 \ . \tag{4.8}$$

Condition (4.8) is an equation to determine the "$x^{(1)}$ behavior" of f and, hence, the $x^{(1)}$ behavior of both $\rho^{(1)}$ and $u^{(0)}$. Equation (4.8) can be expressed in terms of Burgers' equation, with $((1+\gamma)/4)x^{(1)}$ playing the role of the "time" variable and x playing the role of the spacial variable. In particular, the solution to (4.8) which satisfies the "initial" condition that $f = \tilde{f}(t-x)$ when $x^{(1)}=0$ can be expressed as

$$f = \tilde{f}(\tau), \text{ where } \tau = t-x + ((\gamma+1)/2)x^{(1)}\tilde{f}(\tau). \tag{4.9}$$

As an application of these results, we let the (dimensional) fluid velocity at x=0 be $\tilde{u} = \tilde{U}\sin(\omega\tilde{t})$. We then use $\tilde{U}$ as the typical velocity of the flow and let $L = \tilde{c}_0/\omega$ be the typical length associated with the flow. (Note that $k = 1/L = \omega/\tilde{c}_0$ is the linear wave number for this example.) Then the boundary condition for the flow, when expressed in terms of the nondimensional variables (2.4), becomes

$$u = \sin(t), \quad \text{at } x=0 \text{ , for all } t\geq 0 \text{ (boundary condition)}. \tag{4.10}$$

Using the expansions (3.2) in (4.10) we find that the individual coefficients $u^{(k)}$ and $\rho^{(k)}$ satisfy the initial and boundary conditions

$$\left.\begin{array}{r} u^{(k)} = 0 \\ \rho^{(k+1)} = 0 \end{array}\right\} \text{ at } t=0 \text{ (for all } x\geq 0) \text{ for } k = 0, 1, 2, \ldots , \tag{4.11}$$

$$\left.\begin{array}{l} u^{(0)} = \sin t \\ u^{(k)} = 0 \end{array}\right\} \text{ at } x=0 \text{ (for all } t\geq 0) \text{ for } k = 1, 2, 3, \ldots . \tag{4.12}$$

From equations (4.5), we find

$$u^{(0)} \equiv \rho^{(1)} = f(t-x, x^{(1)}, x^{(2)}, \ldots). \tag{4.13}$$

If we were to terminate our perturbation expansion at this point and set $x^{(i)}$ for $i \geq 1$ equal to zero, then the conditions (4.11)-(4.12) would yield the results

$$u^{(0)} \equiv \rho^{(1)} = \begin{cases} \sin(t-x), & \text{for } 0 \leq x \leq t\ , \\ 0 & , \text{ for } x>t\ . \end{cases} \tag{4.14}$$

This, of course, is just the classical wave solution of linear acoustics. However, if we continue the perturbation solution outlined above and use the solution (4.9), we find

$$u^{(0)} = \rho^{(1)} = \begin{cases} \sin(\tau), & \text{for } 0 \leq x \leq t\ , \\ 0\ , & \text{for } x>t\ , \end{cases} \tag{4.15a}$$

where

$$\tau = t-x + ((\gamma+1)/2)x^{(1)}\sin(\tau)\ . \tag{4.15b}$$

With $\rho^{(1)}$ determined from (4.15), we find that the expressions (4.7) for $\rho^{(2)}$ and $u^{(1)}$, which satisfy conditions (4.11) and (4.12), reduce to

$$\begin{aligned} \rho^{(2)} &= ((3-\gamma)/4)\sin^2(t-x), \\ u^{(1)} &= ((\gamma-3)/4)\left(\sin^2(\tau) - \sin^2(t-x)\right), \end{aligned} \tag{4.16}$$

for $0 \leq x \leq t$, with $\rho^{(2)} = u^{(1)} \equiv 0$ for $t>x$.

Combining the expressions we have obtained for $\rho^{(1)}$ and $\rho^{(2)}$, we can write

$$\begin{aligned} \rho &= 1 + M\sin(\tau) + M^2((3-\gamma)/4)\sin^2(t-x) + O(M^3) + O(M^4x), \\ u &= \sin(\tau) + M((\gamma-3)/4)\left(\sin^2(\tau) - \sin^2(t-x)\right) \end{aligned} \tag{4.17}$$

$$+ O(M^2) + O(M^3x),$$

for $0 \le x \le t$, and $\rho=1$ and $u=0$ for $x>t$. Here τ is defined by equation (4.15b).

To illustrate these results, in Figure 1 we have plotted approximations to $(\rho-1)$ as a function of distance from the source at a fixed time. The multiple scales approximations (solid lines) were obtained using the first three terms on the right side of the first equation in (4.17) and are plotted for two different values of M. In Fig. 1(a), M=0.01 and the resulting wave lies close to linear solution (4.14) (short dashed lines), with only a small distortion due to the nonlinearities in the governing equations being visible. In Fig. 1(b), M has been increased to 0.04. For this case, the gradual steepening of the wave as x increases is evident, with the wave apparently approaching an "N-wave" as x increases. An approximation based on Whitham's (1974) first order approximation, obtained using essentially the method of strained coordinates, is very close to (but not identical to) the multiple scales solution, and hence is not shown on these plots. In these figures we have also plotted a solution (circles) to the basic equations (4.1)-(4.2) obtained by purely numerical means. This numerical solution was obtained by writing the basic equations in characteristic form and then using a MacCormack predictor-corrector technique on the resulting equations. (We thank Dr. Willie R. Watson of the NASA Langley Research Center for carrying out these calculations for us.) As the figures illustrate, there is very good agreement between the multiple scales solution and the numerical solution for these values of the Mach number. We shall comment further on these results in section 9.

5. Three-Dimensional Flow - The Vibrating Sphere

We now wish to describe the acoustic field in an inviscid, isentropic fluid outside a spherical region, when the normal component of the fluid velocity is specified on the surface of this region. It is convenient to think of this spherical region in one of two ways. First, we may regard it as an actual sphere, whose surface executes prescribed vibrations, with the boundary conditions applied on the time averaged surface of the sphere. Under this interpretation, the results which follow are extensions of several "classical" problems in acoustics. Alternatively, we may regard the region as a (mathematical) sphere which encloses a bounded region of "complicated" fluid flow. With this interpretation, our results may be applicable to the problem of determining the acoustic field radiated from bounded (perhaps turbulent) fluid flows. In particular, our results will show that only the normal component of the flow velocity needs to be specified on this spherical surface in order to determine the radiated acoustical field.

To begin, we let the origin of a (nondimensional) Cartesian coordinate system (x_1, x_2, x_3) coincide with the average position of the center of the sphere, whose average radius is a constant a. We then introduce spherical coordinates (r, θ, ψ) (see Figure 2(a)) related to (x_1, x_2, x_3) by the relations $x_1 = r\sin(\theta)\cos(\psi)$, $x_2 = r\sin(\theta)\sin(\psi)$, $x_3 = r\cos(\theta)$. (Here all lengths have been nondimensionalized by referring them to the average radius a of the spherical region.) Then, in terms of these coordinates, equations (2.5)-(2.6) become

$$(5.1) \quad \partial\rho/\partial t + M\Big((1/r^2)(\partial/\partial r)(r^2\rho u_r) + (1/r\sin(\theta))(\partial/\partial\theta)(\sin(\theta)\rho u_\theta) + (1/r\sin(\theta))(\partial/\partial\psi)(\rho u_\psi)\Big) = 0,$$

$$(5.2) \quad \partial\rho/\partial r + \rho^{2-\gamma}\Big\{M\partial u_r/\partial t + M^2\Big(u_r(\partial u_r/\partial r) + (u_\theta/r)(\partial u_r/\partial\theta) + (u_\psi/r\sin(\theta))(\partial u_r/\partial\psi) - (u_\theta)^2/r - (u_\psi)^2/r\Big)\Big\} = 0,$$

$$(5.3) \quad \partial\rho/\partial\theta + r\rho^{2-\gamma}\Big\{M\partial u_\theta/\partial t + M^2\Big(u_r(\partial u_\theta/\partial r) + (u_\theta/r)(\partial u_\theta/\partial\theta) + (u_\psi/r\sin(\theta))(\partial u_\theta/\partial\psi) + (u_r u_\theta)/r - (u_\psi)^2\cot(\theta)/r\Big)\Big\} = 0,$$

$$(5.4) \quad \partial\rho/\partial\psi + (r\sin(\theta))\rho^{2-\gamma}\Big\{M\partial u_\psi/\partial t + M^2\Big(u_r(\partial u_\psi/\partial r) + (u_\theta/r)(\partial u_\psi/\partial\theta) + (u_\psi/r\sin(\theta))(\partial u_\psi/\partial\psi) + (u_r u_\psi)/r + (u_\theta u_\psi)\cot(\theta)/r\Big)\Big\} = 0.$$

In (5.1)-(5.4), u_r, u_θ, u_ψ represent the fluid velocity components in the positive r, θ and ψ directions, respectively.

We shall assume that u_r is a specified function, say $V(\theta,\psi,t)$, at r=1 for all $t\geq 0$, and that u_r, u_θ, u_ψ and $\rho-1$ are all specified to be zero at t=0 for all $1<r<\infty$, i.e.

$$(5.5) \qquad u_r\Big|_{\substack{r=1\\ t\geq 0}} = V(\theta,\psi,t), \quad (u_r,u_\theta,u_\psi)\Big|_{\substack{t=0\\ r>1}} = (0,0,0) \text{ and } \rho\Big|_{\substack{t=0\\ r>1}} = 1 .$$

Following the method of multiple scales, we introduce the spacial scales r_i, i = 0, 1, 2, ... , related to r by

$$(5.6) \qquad r_0 = r, \ r_i = M^i r, \ i = 1, 2, \dots .$$

Then, for "small" values of M, we look for solutions for the density and velocity components in the form

$$(5.7) \qquad \rho = 1 + M\rho^{(1)} + M^2\rho^{(2)} + \dots = \sum_{j=0}^{\infty} \rho^{(j)}M^j , \quad \text{with } \rho^{(0)}\equiv 1,$$

$$u_r = u_r^{(0)} + Mu_r^{(1)} + M^2u_r^{(2)} + \dots = \sum_{j=0}^{\infty} u_r^{(j)}M^j,$$

with analogous expressions holding for u_θ and u_ψ. Here each of the coefficient functions $\rho^{(j)}$, $u_r^{(j)}$, etc., is independent of M, but, in general, will depend upon t and the spacial scales r_i, i.e. $\rho^{(j)} = \rho^{(j)}(t,r_0,r_1,r_2,\dots)$, $u_r^{(j)} = u_r^{(j)}(t,r_0,r_1,r_2,\dots)$, ...

To determine these coefficient functions, we substitute the expansions

(5.7) into equations (5.1)-(5.4) and use the relation

$$\partial/\partial r = D_0 + MD_1 + M^2D_2 + \dots , \text{ where } D_j \equiv \partial/\partial r_j . \tag{5.8}$$

We then collect coefficients of like powers of M on the left side of each equation, as described in section 3, and find the following system of equations satisfied by $\rho^{(j)}$, $u_r^{(j)}$, $u_\theta^{(j)}$, and $u_\psi^{(j)}$:

$$\partial\rho^{(1)}/\partial t + (1/r^2)D_0\left(r^2u_r^{(0)}\right) + (1/r\sin(\theta))(\partial/\partial\theta)(\sin(\theta)u_\theta^{(0)}) + (1/r\sin(\theta))\partial u_\psi^{(0)}/\partial\psi = 0, \tag{5.9a}$$

$$D_0\rho^{(1)} + \partial u_r^{(0)}/\partial t = 0, \tag{5.9b}$$

$$\partial\rho^{(1)}/\partial\theta + r\partial u_\theta^{(0)}/\partial t = 0, \tag{5.9c}$$

$$\partial\rho^{(1)}/\partial\psi + r\sin(\theta)\partial u_\psi^{(0)}/\partial t = 0; \tag{5.9d}$$

$$\partial\rho^{(k+1)}/\partial t + (1/r^2)D_0\left(r^2u_r^{(k)}\right) + (1/r\sin(\theta))(\partial/\partial\theta)(\sin(\theta)u_\theta^{(k)}) + (1/r\sin(\theta))\partial u_\psi^{(k)}/\partial\psi = F^{(k)}, \tag{5.10a}$$

$$D_0\rho^{(k+1)} + \partial u_r^{(k)}/\partial t = G_r^{(k)}, \tag{5.10b}$$

$$\partial\rho^{(k+1)}/\partial\theta + r\partial u_\theta^{(k)}/\partial t = G_\theta^{(k)}, \tag{5.10c}$$

$$\partial\rho^{(k+1)}/\partial\psi + r\sin(\theta)\partial u_\psi^{(k)}/\partial t = G_\psi^{(k)}, \text{ for } k=1, 2, \dots . \tag{5.10d}$$

(Equations (5.9) follow from the terms in equations (5.1)-(5.4) which are O(M), while equations (5.10) follow from the terms in these equations which are $O(M^{k+1})$.) Here the functions $F^{(k)}$, $G_r^{(k)}$, $G_\theta^{(k)}$ and $G_\psi^{(k)}$ depend only upon $\rho^{(j)}$ with $j<k+1$ and $u_r^{(j)}$, $u_\theta^{(j)}$, and $u_\psi^{(j)}$ with $j<k$.

In the next sections we shall show how equations (5.9)-(5.10) can be solved recursively.

6. Solution for the lowest order perturbation coefficients

The form of equations (5.9) suggests that we look for a solution for the lowest order perturbation coefficient functions in the form

$$\rho^{(1)} = -\partial\varphi/\partial t,\ u_r^{(0)} = \partial\varphi/\partial r,$$
$$u_\theta^{(0)} = (1/r)\partial\varphi/\partial\theta,\ u_\psi^{(0)} = (1/r\sin(\theta))\partial\varphi/\partial\psi, \tag{6.1}$$

where φ is a smooth function of its arguments, which still needs to be determined. Using equations (6.1), we see that equations (5.9b)-(5.9d) are satisfied for any choice of φ, while equation (5.9a) leads to the requirement that φ must satisfy

$$\partial^2\varphi/\partial t^2 = (1/r^2)(\partial/\partial r)\left(r^2\partial\varphi/\partial r\right)$$
$$+ (1/r^2\sin(\theta))(\partial/\partial\theta)\left(\sin(\theta)\partial\varphi/\partial\theta\right) + (1/r^2\sin^2(\theta))\partial^2\varphi/\partial\psi^2, \tag{6.2}$$

which is just the usual (linear) wave equation. (Here we have used the relation $D_0 \equiv \partial/\partial r_0 = \partial/\partial r$.) This equation has solutions of the form

$$\varphi = F_n(r,t)P_n^m(\cos(\theta))\cos(m\psi-\beta),\quad \text{for } n = 0, 1, 2, \dots,\ \text{and } m = 0, 1, \dots, n, \tag{6.3}$$

where P_n^m is the associated Legendre polynomial, β is an arbitrary constant, and F_n satisfies the differential equation

$$\partial^2F_n/\partial t^2 = (1/r^2)(\partial/\partial r)\left(r^2\partial F_n/\partial r\right) - (1/r^2)n(n+1)F_n.$$

The outgoing solution for F_n can be expressed as

$$(6.4) \qquad F_n(r,t) = \sum_{j=1}^{n+1} a_{n,j} f^{(n+1-j)}(t-r+1)/r^j,$$

$$a_{n,1} = -1, \quad a_{n,j+1} = \Big((n+1-j)(j+n)/(2j)\Big)a_{n,j}, \quad j = 1,\ 2,\ \dots,\ n,$$

where f is an arbitrary (smooth) function of its argument.

If we now apply the boundary condition (5.5)

$$V(\theta,\psi,t) \equiv u_r(r,\theta,\psi,t)\Big|_{r=1} = \sum_{j=0}^{\infty} M^j u_r^{(j)}(r_0,\ r_1,\ r_2,\ \dots,\theta,\psi,t)\Big|_{r=1},$$

we see that we can set

$$(6.5) \qquad \begin{aligned} u_r^{(0)}\Big|_{r=1} &= V(\theta,\psi,t), \text{ for } j=0, \\ u_r^{(j)}\Big|_{r=1} &= 0 \ , \text{ for } j\geq 1 \ . \end{aligned}$$

We shall assume that V can be expressed as (a linear combination of terms of the form)

$$(6.6) \qquad V(\theta,\psi,t) = Q(t)P_n^m(\cos(\theta))\cos(m\psi-\beta),$$

where Q(t) is a specified function of t, and n and m are non-negative integers, with $0\leq m\leq n$. Using this expression for V, along with the condition (6.5) on $u_r^{(0)}$, as well as the expressions (6.3) for φ and (6.1) for $u_r^{(0)}$, we find that f must satisfy the condition

$$(6.7) \qquad f^{(n+1)}(t) - \sum_{j=1}^{n}\Big((j(j+1)+n(n+1))/(2j)\Big)a_{n,j} f^{(n+1-j)}(t)$$

$$- (n+1)a_{n,n+1}f(t) = Q(t) .$$

Equation (6.7) is an inhomogeneous, linear ordinary differential equation of order n+1, with constant coefficients, for the unknown function f(t). For a few small values of n this equation becomes

$$\begin{aligned}
&n=0: && f' + f = Q(t);\\
&n=1: && f'' + 2f' + 2f = Q(t);\\
(6.8)\quad &n=2: && f''' + 4f'' + 9f' + 9f = Q(t);\\
&n=3: && f'''' + 7f''' + 27f'' + 60f' + 60f = Q(t);\\
&n=4: && f''''' + 11f'''' + 65f''' + 240f'' + 525f' + 525f = Q(t).
\end{aligned}$$

Once f has been determined, we can express the solution for φ as

$$(6.9)\qquad \varphi = \left\{ \sum_{j=1}^{n+1} a_{n,j} f^{(n+1-j)}(t')/r^j \right\} P_n^m(\cos(\theta))\cos(m\psi-\beta),$$

where $t' \equiv t-r+1$. In general, f will also depend on the "slow" spacial scales r_1, r_2, ..., although this dependence has not been denoted explicitly.

As an illustration of these results, we consider the special case when $Q(t) = \sin(\omega t)$. Then the steady state (periodic) solution to (6.7) (which can be viewed mathematically as a particular solution to this equation) can be expressed as

$$(6.10)\qquad f(t) = A\sin(\omega t-\alpha), \text{ when } Q(t) = \sin(\omega t),$$

where, for a few small values of n, A and α are given by

$$\begin{aligned}
&n=0: && A = (1+\omega^2)^{-1/2},\ \alpha = \tan^{-1}(\omega),\\
&n=1: && A = (4+\omega^4)^{-1/2},\ \alpha = \tan^{-1}\left(2\omega/(2-\omega^2)\right),
\end{aligned}$$

$$n=2: \quad A = (81+9\omega^2-2\omega^4+\omega^6)^{-1/2}, \ \alpha = \tan^{-1}\left(\omega(9-\omega^2)/(9-4\omega^2)\right),$$

$$n=3: \quad A = (3600+360\omega^2+9\omega^4-5\omega^6+\omega^8)^{-1/2},$$

$$\alpha = \tan^{-1}\left(\omega(60-7\omega^2)/(\omega^4-27\omega^2+60)\right),$$

$$n=4: \quad A = (275625+23625\omega^2+900\omega^4-5\omega^6-9\omega^8+\omega^{10})^{-1/2},$$

$$\alpha = \tan^{-1}\left(\omega(525-65\omega^2+\omega^4)/(525-240\omega^2+11\omega^4)\right).$$

Solutions to the homogeneous version of equation (6.7) are of the form $f(t) = e^{\lambda t}$, where the real part of λ is negative. Hence the corresponding solutions for $f(t)$ are not periodic in t and, in fact, they all decay to zero as time increases.

Before we consider the higher order perturbation coefficients, we can use the results above to calculate a first approximation to the radial component I of the sound intensity, defined by

$$I = \rho_0 c_0^2 U\langle M\rho \bullet u_r\rangle, \tag{6.11}$$

where the symbol "<g>" denotes the time averaged value of the quantity g. Using the expressions (6.1), (6.3) and (6.4) above we find, for the case when Q is given as in (6.10), that

$$I = \rho_0 c_0 U^2 \omega^{2m+2} A^2 \left(P_n^m(\cos(\theta))\cos(m\psi-\beta)\right)^2 / 2r^2 + O(1/r^3), \text{ as } r\to\infty.$$

For $\omega \equiv ka \ll 1$, we find that the *maximum* intensity of I, which we denote by $\tilde{I}_n$, for different values of n, is given by

$$n=0: \quad \tilde{I}_0 = \left(\rho_0 c_0 U^2 (ka)^2/2\right)(a/\hat{r})^2 + O((ka)^4/\hat{r}^2) + O(1/r^3);$$

$$n=1: \quad \tilde{I}_1 = \left(\rho_0 c_0 U^2 (ka)^2/2\right)(a/\hat{r})^2\left((ka)^2/2^2\right) + O((ka)^6/\hat{r}^2) + O(1/r^3);$$

$$n=2: \quad \tilde{I}_2 = \left(\rho_0 c_0 U^2 (ka)^2/2\right)(a/\hat{r})^2\left((ka)^4/3^2\right) + O((ka)^8/\hat{r}^2) + O(1/r^3).$$

In general, for any positive integer n, the ratio of the maximum intensity $\tilde{I}_n$ to the maximum intensity $\tilde{I}_0$ of the pulsating sphere approaches $(ka)^{2n}/(n+1)^2$, as ka approaches zero.

7. Solutions for the higher order perturbation coefficients

Equations (5.10) with k=1 can be written as

(7.1a)
$$\partial\rho^{(2)}/\partial t + (1/r^2)D_0\left(r^2 u_r^{(1)}\right) + (1/r\sin(\theta))(\partial/\partial\theta)(\sin(\theta)u_\theta^{(1)}) + (1/r\sin(\theta))\partial u_\psi^{(1)}/\partial\psi = -\Big((1/r^2)D_0(r^2\rho^{(1)}u_r^{(0)}) + D_1 u_r^{(0)} + (1/r\sin(\theta))(\partial/\partial\theta)(\sin(\theta)\rho^{(1)}u_\theta^{(0)}) + (1/r\sin(\theta))(\partial/\partial\psi)(\rho^{(1)}u_\psi^{(0)})\Big),$$

(7.1b)
$$D_0\rho^{(2)} + \partial u_r^{(1)}/\partial t = -\Big(u_r^{(0)}D_0 u_r^{(0)} + (2-\gamma)\rho^{(1)}\partial u_r^{(0)}/\partial t + D_1\rho^{(1)} + (u_\theta^{(0)}/r)(\partial u_r^{(0)}/\partial\theta) + (u_\psi^{(0)}/r\sin(\theta))(\partial u_r^{(0)}/\partial\psi) - (u_\theta^{(0)})^2/r - (u_\psi^{(0)})^2/r\Big),$$

(7.1c)
$$\partial\rho^{(2)}/\partial\theta + r\partial u_\theta^{(1)}/\partial t = -r\Big((2-\gamma)\rho^{(1)}\partial u_\theta^{(0)}/\partial t + u_r^{(0)}D_0 u_\theta^{(0)} + (u_\theta^{(0)}/r)(\partial u_\theta^{(0)}/\partial\theta) + (u_\psi^{(0)}/r\sin(\theta))(\partial u_\theta^{(0)}/\partial\psi) + (u_r^{(0)}u_\theta^{(0)})/r - (u_\psi^{(0)})^2\cot(\theta)/r\Big),$$

(7.1d)
$$\partial\rho^{(2)}/\partial\psi + r\sin(\theta)\partial u_\psi^{(1)}/\partial t =$$

$$-(r\sin(\theta))\Big((2-\gamma)\rho^{(1)}\partial u_\psi^{(0)}/\partial t + u_r^{(0)}D_0 u_\psi$$

$$+ (u_\theta^{(0)}/r)(\partial u_\psi^{(0)}/\partial\theta) + (u_\psi^{(0)}/r\sin(\theta))(\partial u_\psi^{(0)}/\partial\psi)$$

$$+ (u_r^{(0)}u_\psi^{(0)})/r + (u_\theta^{(0)}u_\psi^{(0)})\cot(\theta)/r\Big).$$

To begin our construction of solutions to equations (7.1), we first examine the behavior of the right sides of these equations as $r\to\infty$. Using equations (6.1), (6.3), and (6.4) we find

$$\rho^{(1)} = -\partial\varphi/\partial t = (1/r)f^{(n+1)}P_n^m(\cos(\theta))\cos(m\psi-\beta) + O(1/r^2),$$

$$(7.2)\quad u_r^{(0)} = \partial\varphi/\partial r = (1/r)f^{(n+1)}P_n^m(\cos(\theta))\cos(m\psi-\beta) + O(1/r^2),$$

$$u_\theta^{(0)} = (1/r)\partial\varphi/\partial\theta = (1/r^2)f^{(n)}P_n^{m\prime}(\cos(\theta))(\sin(\theta))\cos(m\psi-\beta) + O(1/r^3),$$

$$u_\psi^{(0)} = (1/r\sin(\theta))\partial\varphi/\partial\psi$$

$$= (m/r^2\sin(\theta))f^{(n)}P_n^m(\cos(\theta))\sin(m\psi-\beta) + O(1/r^3), \quad \text{as } r\to\infty.$$

(Here it is understood that the argument of f is $t'=t-r+1$.) Then, using these expressions, we can write equations (7.1) as

$$(7.3a)\quad \partial\rho^{(2)}/\partial t + (1/r^2)D_0\Big(r^2u_r^{(1)}\Big)$$

$$+ (1/r\sin(\theta))(\partial/\partial\theta)(\sin(\theta)u_\theta^{(1)}) + (1/r\sin(\theta))\partial u_\psi^{(1)}/\partial\psi =$$

$$-D_1u_r^{(0)} + (2/r^2)f^{(n+1)}f^{(n+2)}\Big(P_n^m(\cos(\theta))\cos(m\psi-\beta)\Big)^2 + O(1/r^3),$$

$$(7.3b)\quad D_0\rho^{(2)} + \partial u_r^{(1)}/\partial t =$$

$$-D_1\rho^{(1)} + ((\gamma-1)/r^2)f^{(n+1)}f^{(n+2)}\Big(P_n^m(\cos(\theta))\cos(m\psi-\beta)\Big)^2 + O(1/r^3),$$

$$(7.3c)\quad \partial\rho^{(2)}/\partial\theta + r\partial u_\theta^{(1)}/\partial t =$$

$$((1-\gamma)/r^2)\left(f^{(n+1)}\right)^2 P_n^m(\cos(\theta))P_n^{m\prime}(\cos(\theta))\sin(\theta)\cos^2(m\psi-\beta) + O(1/r^3),$$

$$\text{(7.3d)} \quad \partial\rho^{(2)}/\partial\psi + r\sin(\theta)\partial u_\psi^{(1)}/\partial t =$$

$$m((\gamma-1)/r^2)\left(f^{(n+1)}P_n^m(\cos(\theta))\right)^2 \cos(m\psi-\beta)\sin(m\psi-\beta) + O(1/r^3).$$

We now examine the effect on the solutions $\rho^{(2)}$, $u_r^{(1)}$, etc., of each of the terms on the right sides of equations (7.3). In particular, we must first examine the behavior of the terms $D_1\rho^{(1)}$ and $D_1u_r^{(0)}$ as $r\to\infty$. Now, if we assume that $D_1\rho^{(1)}$ and $D_1u_r^{(0)}$ are $O(1/r)$ as $r\to\infty$, then these terms will give rise to terms in $\rho^{(2)}$ and $u_r^{(1)}$ which are $O(\log(r))$ as $r\to\infty$. However, since $\rho^{(2)}$ must decay at least as fast as $\rho^{(1)}$, which is $O(1/r)$ as $r\to\infty$, we see that this cannot be allowed. Consequently, we shall now require that $D_1\rho^{(1)}$ and $D_1u_r^{(0)}$ are $O(1/r^2)$ as $r\to\infty$, and we will show that this assumption allows us to determine the quantities of interest in a consistent manner. Using (7.2), we can express this assumption as

$$\begin{aligned} D_1\rho^{(1)} &= D_1u_r^{(0)} + O(1/r^3) \\ &= (1/r^2)\left(rD_1f^{(n+1)}\right)P_n^m(\cos(\theta))\cos(m\psi-\beta) + O(1/r^3), \end{aligned}$$

where $rD_1f^{(n+1)}$ is bounded as $r\to\infty$. Also, as we shall show below, it is convenient to think of the argument t' of f as being replaced by a new argument, say, $\tau = \tau(t-r+1,\theta,\psi,r_1,r_2,\ldots,M)$, which must be determined. Then we can write

$$\text{(7.4a)} \qquad D_1f^{(n+1)}(\tau) = f^{(n+2)}(\tau)(D_1\tau).$$

Furthermore, the angular dependence of the terms on the right sides of equations (7.3) suggests that $D_1\tau$ should be proportional to $P_n^m(\cos(\theta))\cdot\cos(m\psi-\beta)$. Thus, we write

$$D_1\tau = h(\tau)P_n^m(\cos(\theta))\cos(m\psi-\beta), \tag{7.4b}$$

where $h(\tau)$ must be determined. With these assumptions, we see that all of the terms on the right side of (7.3) are $O(1/r^2)$ (at least) as $r\to\infty$, as required. Thus, we can express the (particular) solutions for $\rho^{(2)}$, $u_r^{(1)}$, $u_\theta^{(1)}$, and $u_\psi^{(1)}$ in the form

$$\begin{aligned}
\rho^{(2)} &= \left\{(1/r)\log(r)F_0' - (1/r^2)\log(r)F_0 + (1/r^2)F_1\right\} \\
&\qquad\cdot\left(P_n^m(\cos(\theta)\cos(m\psi-\beta)\right)^2 + O(1/r^3), \\
u_r^{(1)} &= \left\{(1/r)\log(r)F_0' + (1/r^2)F_2\right\} \\
&\qquad\cdot\left(P_n^m(\cos(\theta)\cos(m\psi-\beta)\right)^2 + O(1/r^3), \\
u_\theta^{(1)} &= \left\{(1/r^2)\log(r)F_0 + (1/r^3)F_3\right\} \\
&\qquad\cdot 2P_n^m(\cos(\theta))P_n^{m\,\prime}(\cos(\theta))\sin(\theta)\cos^2(m\psi-\beta) + O(1/r^4), \\
u_\psi^{(1)} &= \left\{(1/r^2\sin(\theta))\log(r)F_0 + (1/r^3\sin(\theta))F_4\right\} \\
&\qquad\cdot 2m\left(P_n^m(\cos(\theta))\right)^2\cos(m\psi-\beta)\sin(m\psi-\beta) + O(1/r^4),
\end{aligned} \tag{7.5}$$

where each $F_j = F_j(\tau,r)$ is a function to be determined and the primes denote differentiation with respect to τ. Substituting the expressions (7.5) into equations (7.3), and using (7.4), we obtain the relations:

(7.6a) $$F_0' = ((\gamma+1)/2)f^{(n+1)}f^{(n+2)} - rf^{(n+2)}h(\tau) + O(1/r),$$

(7.6b) $$F_1' = F_2' + ((3-\gamma)/2)f^{(n+1)}f^{(n+2)} + O(1/r),$$

(7.6c) $$F_3' = F_2 + ((7-3\gamma)/2)\left(f^{(n+2)}\right)^2 + O(1/r),$$

(7.6d) $$F_4' = F_2 + ((7-\gamma)/2)\left(f^{(n+1)}\right)^2 + O(1/r).$$

In (7.6), the function F_2 can be determined by examining the terms in equations (7.3) that are $O(1/r^3)$. From the form of the solution (7.5), we see that we must demand that $F_0 = O(1/\log(r))$, as $r\to\infty$, since otherwise $\rho^{(2)}$ will not decay as fast as the $O(1/r)$ rate of decay of $\rho^{(1)}$. Using this condition, we see that the right side of equation (7.6a) must be $O(1/\log(r))$, as $r\to\infty$. Using the relation (7.4b), this condition can be written as

(7.7) $$D_1(g) + ((\gamma+1)/4)P_n^m(\cos(\theta))\cos(m\psi-\beta)(1/r^2)D_0(r^2g^2) = O(1/(r^2\log(r))),$$

as $r\to\infty$, where $g \equiv f^{(n+1)}/r$. (Here we have assumed that $D_0\tau = -1 + O(1/\log(r))$ and $\partial\tau/\partial t = 1 + O(1/\log(r))$, as $r\to\infty$. See equations (7.9) below.)

Equation (7.7) is the major result of this section and it serves to determine the behavior of $\rho^{(1)}$, $u_r^{(0)}$, $u_\theta^{(0)}$, and $u_\psi^{(0)}$ as functions of r_1. It can be interpreted as a "spherical" Burgers' equation in the "time-like" variable $((\gamma+1)/2)P_n^m(\cos(\theta))\cos(m\psi-\beta)r_1$. In particular, we can express the solution of (7.7) which satisfies the "initial" condition $g = f^{(n+1)}(t+1-r)/r$ when M=0 as

(7.8a) $$g = f^{(n+1)}(\tau)/r, \text{ where}$$

(7.8b) $$\tau = t-r+1 + ((\gamma+1)/2)\tilde{F}_n(\tau,r_1)P_n^m(\cos(\theta))\cos(m\psi-\beta),$$

$$\tilde{F}_n(\tau,r_1) \equiv -\sum_{j=1}^{n+1} a_{n,j} f^{(n+2-j)}(\tau)M^j\int_M^{r_1}(1/s)^j ds$$

$$= Mf^{(n+1)}(\tau)\log(r_1/M) + M\sum_{j=2}^{n+1} \tilde{a}_{n,j} f^{(n+2-j)}(\tau)\Big((M/r_1)^{j-1}-1\Big),$$

where $\tilde{a}_{n,j} = a_{n,j}/(j-1)$, $j = 2, 3, \ldots, n+1$.

Thus, both the r_0 and r_1 behavior of $\rho^{(1)}$, $u_r^{(0)}$, $u_\theta^{(0)}$, and $u_\psi^{(0)}$ are completely determined by equations (6.9), (6.1), with t' replaced by τ, and (7.8b).

Before we consider some specific applications of these results, we note that, from its definition in (7.8b), τ has the following properties:

(7.9a) $\tau \to t-r+1$, as $M\to 0$;

(7.9b) $\tau \to t-r+1$, as $r\to 1$;

(7.9c) $\partial\tau/\partial t = 1 + O(1/\log(r))$, $\partial\tau/\partial r = -1 + O(1/\log(r))$, as $r\to\infty$;

(7.9d) $\partial\tau/\partial\theta = O(M)$, and $\partial\tau/\partial\psi = O(M)$, as $M\to 0$;

and

(7.9e) $$D_1\tau = ((\gamma+1)/2)P_n^m(\cos(\theta))\cos(m\psi-\beta)f^{(n+1)}(\tau)/r + O(1/r^2), \text{ as } r\to\infty.$$

Properties (7.9a) and (7.9b) insure that the linear solution is recovered, either as $M\to 0$ or as we approach the surface of the sphere. Property (7.9c) shows that our definition of τ is consistent with the assumptions made above, while property (7.9d) shows that the angular variation of τ is "small" and, hence, will be described by the next level of the perturbation expansion. Property (7.9e) shows that $D_1\tau$ is $O(1/r)$, as $r\to\infty$, and hence τ satisfies the requirement that $rD_1\tau$ is bounded as $r\to\infty$. We shall comment further on the properties of τ in section 9.

8. Applications

a) The Pulsating Sphere

As our first application, we consider the pulsating sphere (see Figure 2(b)). In this case, the (nondimensional) radial component of the velocity of the surface of the sphere is given by $V = \sin(\omega t)$ and hence we set n=0 and m=0 in our results from sections 6 and 7. Then the steady state solution for f(t) is given by (6.10) with n=0, and φ is given by (6.9) with n=0, i.e.

$$f(t) = (1+\omega^2)^{-1}\Big(\sin(\omega t) - \omega\cos(\omega t)\Big) \quad \text{and} \quad \varphi = -f(\tau)/r, \tag{8.1}$$

where τ is determined by (7.8b) with n=m=0, i.e.,

$$\tau = t-r+1 + ((\gamma+1)/2)Mf'(\tau)\log(r_1/M). \tag{8.2}$$

Using these definitions in equations (6.1) and (7.1) we find for this case

$$\rho^{(1)} = \Big(\omega/((1+\omega^2)r)\Big)(\cos(\omega\tau) + \omega\sin(\omega\tau)),$$

$$\begin{aligned}\rho^{(2)} = {} & 2b_3\omega\cos(2\omega\tau)/r - 2b_2\omega\sin(2\omega\tau)/r \\ & - 1/(4r^4(1+\omega^2)) + (1-\gamma)\omega^2/(4r^2(1+\omega^2)) \\ & + \Big((17+\gamma)\omega^2/(16r^3(1+\omega^2)^2) - (13-3\gamma)(\omega^2-1)/(64r^4(1+\omega^2)^2) \\ & \qquad + (5+\gamma)\omega^2(\omega^2-1)/(8r^2(1+\omega^2)^2)\Big)\cos(2\omega\tau) \\ & + \Big((13-3\gamma)\omega/(32r^4(1+\omega^2)^2) - (5+\gamma)\omega^3/(4r^2(1+\omega^2)^2) \\ & \qquad + (17+\gamma)\omega(\omega^2-1)/(32r^3(1+\omega^2)^2)\Big)\sin(2\omega\tau) \\ & + O(1/r^5),\end{aligned} \tag{8.3}$$

and

$$b_2 = -3\omega^2(14-2\gamma+17\omega^2+\gamma\omega^2)/(16(1+\omega^2)^2(1+4\omega^2)),$$

$$b_3 = \omega(29-3\gamma-17\omega^2+15\gamma\omega^2-64\omega^4)/(32(1+\omega^2)^2(1+4\omega^2)).$$

(The corresponding expressions for the perturbation coefficient functions $u_r^{(0)}$ and $u_r^{(1)}$ for the radial component of the velocity are listed in the appendix.)

In Figure 3, we have plotted $r(\rho^{(1)} + M\rho^{(2)})$ (solid lines) as a function of r at a fixed time for ω=1.25 and for two different values of M. Here it is convenient to think of $\rho^{(1)} + M\rho^{(2)}$ as our approximation to the normalized, nondimensional acoustical density field $(\rho-1)/M$ (from equation (3.2)). In Figure 3 we have also plotted the approximation to $r(\rho-1)/M$ based on Whitham's (1974) first order solution (long dashed lines), the classical linear solution (short dashed lines), and a solution obtained by purely numerical methods (circles). This numerical solution was obtained using a MacCormack predictor-corrector scheme on equations (5.1)-(5.4). As Figure 3 illustrates, there is very good agreement between the multiple scales perturbation solutions and the numerical solutions, even for a Mach number of 0.3, where the gradual steepening and asymmetry of the wave is apparent.

b) The Oscillating Sphere

As a second application, we consider the oscillating sphere (see Figure 2(c).) In this case, the radial component of the velocity of the surface of the sphere is given by $V = \cos(\theta)\sin(\omega t)$ and, hence, we set n=1 and m=0 in our results from sections 6 and 7. Then the steady state solution for f(t) is given by (6.10) with n=1, and φ is given by (6.9) with n=1, i.e.

$$f(t) = (4+\omega^4)^{-1}\Big((2-\omega^2)\sin(\omega t) - 2\omega\cos(\omega t)\Big), \tag{8.4}$$

$$\varphi = -\cos(\theta)\Big(f'(\tau)/r + f(\tau)/r^2\Big),$$

where τ is now determined by (7.8b) with n=1 and m=0, i.e.

$$\tau = t-r+1 + ((\gamma+1)/2)M\left\{f''(\tau)\log(r_1/M) - f'(\tau)\left((M/r_1)-1\right)\right\}\cos(\theta). \tag{8.5}$$

Using these expressions in equations (6.1) and (7.1), and using Wolfram's (1991) symbolic computation system *Mathematica*, we find for this case

$$\rho^{(1)} = \left(\omega(2-\omega^2+2r\omega^2)\cos(\omega\tau)/r^2 + \omega^2(2-2r+r\omega^2)\sin(\omega\tau)/r^2\right)\cos(\theta)/(4+\omega^4),$$

$$\begin{aligned}
\rho^{(2)} = \Big\{&2\omega(f_{0s}\cos(2\omega\tau)-f_{0c}\sin(2\omega\tau))/r - (3\omega/2)(f_{2s}\cos(2\omega\tau)+f_{2c}\sin(2\omega\tau))/r^3 \\
&+ 3\omega^2(f_{2c}\cos(2\omega\tau)+f_{2s}\sin(2\omega\tau))/r^2 + 2\omega^3(f_{2s}\cos(2\omega\tau)-f_{2c}\sin(2\omega\tau))/r \\
&+ (1-\gamma)\omega^2(4+\omega^4)/8r^4 + (1-\gamma)\omega^4(4+\omega^4)/8r^2 \\
&+\Big((323+3\gamma)\omega^2(2-\omega^2)/64r^5 - (7+\gamma)\omega^4(2-\omega^2)/2r^3 \\
&\quad - (259+19\gamma)\omega^2(4-8\omega^2+\omega^4)/192r^4 + (5+\gamma)\omega^4(4-8\omega^2+\omega^4)/16r^2\Big)\cos(2\omega\tau) \\
&+\Big((5+\gamma)\omega^5(2-\omega^2)/4r^2 - (259+19\gamma)\omega^3(2-\omega^2)/48r^4 \\
&\quad + (7+\gamma)\omega^3(4-8\omega^2+\omega^4)/8r^3 \\
&\quad - \omega(323+3\gamma)(4-8\omega^2+\omega^4)/256r^5\Big)\sin(2\omega\tau)\Big\}/(4+\omega^4)^2 \\
&+ \cos(2\theta)\Big\{(9\omega/2)(-f_{2s}\cos(2\omega\tau)+f_{2c}\sin(2\omega\tau))/r^3 \\
&- 9\omega^2(f_{2c}\cos(2\omega\tau)+f_{2s}\sin(2\omega\tau))/r^2 + 6\omega^3(f_{2s}\cos(2\omega\tau)-f_{2c}\sin(2\omega\tau))/r \\
&+ (3-\gamma)\omega^2(4+\omega^4)/8r^4 + ((1-\gamma)\omega^4(4+\omega^4)/8r^2 \\
&+ \Big(3(33+\gamma)\omega^2(2-\omega^2)/32r^5 - (17+\gamma)\omega^2(4-8\omega^2+\omega^4)/16r^4 \\
&\qquad - 3(9+\gamma)\omega^4(2-\omega^2)/8r^3 + (5+\gamma)\omega^4(4-8\omega^2+\omega^4)/16r^2\Big)\cos(2\omega\tau) \\
&+ \Big((5+\gamma)\omega^5(2-\omega^2)/4r^2 - (17+\gamma)\omega^3(2-\omega^2)/4r^4 \\
&\qquad + 3(9+\gamma)\omega^3(4-8\omega^2+\omega^4)/32r^3 +
\end{aligned}$$

$$- 3\omega(33+\gamma)(4-8\omega^2+\omega^4)/128r^5\Big)\sin(2\omega\tau)\Big\}/(4+\omega^4)^2 + O(1/r^5).$$

The constants f_{0s}, f_{0c}, f_{2s}, and f_{2c} which appear in the definition of $\rho^{(2)}$ are related to the homogeneous solution to equations (7.1) and are defined in the Appendix. (The form of the solutions for $u_r^{(0)}$, $u_\theta^{(0)}$, $u_r^{(1)}$, and $u_\theta^{(1)}$ are also presented in the Appendix.)

The form of these expressions allows us to make an interesting observation concerning the angular dependence of the sound radiated from an oscillating sphere. Specifically, it is interesting to examine the density and velocity in the $\theta=\pi/2$ direction. In this direction, both $\rho^{(1)}$ and $u_r^{(0)}$ are *zero*, while $u_\theta^{(0)}$ is nonzero. *In contrast*, for this direction both $\rho^{(2)}$ and $u_r^{(1)}$ are *nonzero*, while $u_\theta^{(1)}$ is zero (since it contains a factor of $\sin(2\theta)$). Thus, there is a "small" ($O(M^2)$) amount of sound radiated in this direction, which has a characteristic frequency of 2ω. In particular, in the far field we can use the expressions above to write

$$\begin{aligned}\rho^{(2)} &= u_r^{(1)} + O(1/r^2)\\ &= (2\omega/r)\Big((2f_{2s}\omega^2-f_{0s})\cos(2\omega\tau) + (f_{0c}-2f_{2c}\omega^2)\sin(2\omega\tau)\Big) + O(1/r^2),\\ &\qquad \text{along } \theta=\pi/2.\end{aligned}$$

In addition, we find that the expressions for $u_r^{(1)}$ and $u_\theta^{(1)}$ correspond to a *rotational* flow, even though $u_r^{(0)}$ and $u_\theta^{(0)}$ correspond to an *irrotational* flow. The presence of vorticity in the flow can be interpreted as being due to tangential acceleration of the fluid at the boundary, as discussed by Morton (1984). This vorticity appears only very near the sphere and is given by

$$\vec{\nabla}\times\vec{u} = \vec{i}_{\psi}M\sin(2\theta)\Big\{\Big((\gamma+1)\omega^4/(4(4+\omega^4))\Big)(\log(r))/r^3$$
$$+ \Big(\omega^4(4-8\omega^2+\omega^4)/(4+\omega^4)^2\Big)\cos(2\omega\tau)/r^3$$
$$+ \Big(4\omega^5(2-\omega^2)/(4+\omega^4)^2\Big)\sin(2\omega\tau)/r^3 + O(1/r^4)\Big\}.$$

In Figure 4, we have plotted approximations to $r(\rho-1)/(M\cos(\theta))$ as a function of r at a fixed time for $\theta=0$ and $\theta=\pi/3$, as well as $r(\rho-1)/M^2$ at $\theta=\pi/2$, with $\omega=1.25$ and for two different values of M. These approximations are the multiple scales solution $r(\rho^{(1)}+M\rho^{(2)})/\cos(\theta)$ (solid lines), the classical linear solution (short dashed lines), and a solution obtained by purely numerical means (circles). For this case, we found that numerical approximations based on a Jameson type Runge-Kutta finite volume scheme, which is second order accurate in both space and time, gave better agreement with the multiple scales solution than numerical approximations based on the MacCormack scheme, as in the case of the pulsating sphere. (We thank Mr. David Lockard and Dr. Kenneth Brentner of the NASA Langley Research Center for carrying out these calculations for us.) Again, the agreement between the multiple scales solutions and the numerical solutions is qualitatively good.

c) <u>A "Squishing" Sphere</u>

As a third example, we consider the case when the radial component of the surface velocity of the sphere is given by $V = \sin^2(\theta)\sin(2\psi)\sin(\omega t)$ (see Figure 2(d)). Thus, we set n=2 and m=2 in our results of sections 6 and 7, and find from (6.10) with n=2 that the steady solution for f(t) is given by

$$(8.7)\qquad f(t) = (3(81+9\omega^2-2\omega^4+\omega^6))^{-1}\Big((9-4\omega^2)\sin(\omega t) - \omega(9-\omega^2)\cos(\omega t)\Big),$$

while the solution for φ is given by (6.9) with n=2, i.e.,

$$(8.8)\qquad \varphi = -\Big(f''(\tau)/r + 3f'(\tau)/r^2 + 3f(\tau)/r^3\Big)3\sin^2(\theta)\sin(2\psi),$$

with τ determined by (7.8b) with n=2, i.e.,

$$\tau = t-r+1 + ((\gamma+1)/2)\tilde{F}_2(\tau,r_1)3\sin^2(\theta))\sin(2\psi),$$

$$\tilde{F}_2(\tau,r_1) = M\Big\{f'''(\tau)\log(r_1/M) - 3f''(\tau)\Big((M/r_1)-1\Big)$$

$$- (3/2)f'(\tau)\Big((M/r_1)^2-1\Big)\Big\}.$$

Using these expressions we find

$$\rho^{(1)} = \sin(2\psi)\sin^2(\theta)\Big($$
$$((27\omega-12\omega^3)/r^3+(27\omega^3-3\omega^5)/r^2-(9\omega^3-4\omega^5)/r)\cos(\omega\tau) +$$
$$((27\omega^2-3\omega^4)/r^3-(27\omega^2-12\omega^4)/r^2-(9\omega^4-\omega^6)/r)\sin(\omega\tau)\Big)/(81+9\omega^2-2\omega^4+\omega^6),$$

while expressions for $u_r^{(0)}$, $u_\theta^{(0)}$, and $u_\psi^{(0)}$, as well as the *form* of the next order perturbation coefficient functions, are given in the Appendix.

In Figure 5 we have plotted approximations to $r(\rho-1)/(M\sin^2(\theta)\sin(2\psi))$ as a function of r at a fixed value of time with M=0.6 for $\psi = \pi/12$, $\pi/6$, and $\pi/4$ at (a) $\theta=\pi/4$ and (b) $\theta=\pi/2$, using the multiple scales solution $r\rho^{(1)}/(\sin^2(\theta)\sin(2\psi))$ (solid lines) and the classical linear solution (dashed lines). Due to the excessive storage requirements for this fully three-dimensional, time dependent problem, no numerical approximations were computed.

9. Conclusions and Discussion

In this section we shall first make some general observations concerning the perturbation approach we have employed to the problem of sound generation by vibrating bodies, and then comment specifically on the insights gained from the examples we have presented.

We note first that the multiple scales perturbation approach has allowed us to "capture" many of the salient nonlinear features of acoustic wave motion. For example, as the Mach number increases, the nonlinear characteristics of the gradual steepening of the waves, the asymmetries due to "second harmonic" terms, as well as the convergence of the wave to an "N-wave" profile, are all evident in the examples we have considered. These phenomena, of course, are not predicted by linear theory. Thus, it appears that the method has allowed us to obtain approximate analytical solutions which are valid for (much) larger values of M than are the classical linear acoustic solutions. In addition, the solutions assume a rather simple form, being essentially of the same form as the classical linear solutions, but with a different argument. That is, the classical retarded time t+1−r in the linear solution has simply been replaced by τ, where τ is defined implicitly. Other perturbation approaches to this problem have resulted in more involved expressions for the quantities of physical interest (cp., e.g., Crow (1970)). (We should note that, if the present method of analysis were to be continued to a higher order than that presented here, various terms of the form $M^j\left(\log(M)\right)^k$ should be added to the expansions (5.7). However, since the present analysis was terminated at terms which are $O(M^2)$, these terms were not needed and, hence, were not shown explicitly.)

The general form of the equation for τ also provides an insight into the interplay between the effect of the Mach number and the angular variation of the acoustical source on the form of the acoustic wave. For example, if M=0 *or* if the term $P_n^m(\cos(\theta))\cos(m\psi-\beta)$ is zero (corresponding, say, to a particular value of θ or ψ), then from (7.8b) we see that $\tau=t-r+1$ and, hence, our solution (to leading order) reduces just to the linear acoustic solution. However, if $M\neq0$, but the angular term $P_n^m(\cos(\theta))\cos(m\psi-\beta)$ *is zero*, then our solution (to leading order) again reduces to the linear solution and hence the nonzero Mach number has only a second order (i.e. $O(M^2)$) effect on the form of the solution. These observations motivate us to define an "effective" Mach number $M_{eff.}$ by

$$M_{eff.} \equiv MP_n^m(\cos(\theta))\cos(m\psi-\beta).$$

Thus, it is the (angularly dependent) effective Mach number $M_{eff.}$, and not just M, which determines the magnitude of departure of the wave from its linear form. This phenomena is illustrated in Figure 4 for the example of the oscillating sphere, for which n=1 and m=0 in the formulas above, and also in Figure 5, corresponding to n=2 and m=2. In Figure 4(a), for example, the linear form of the wave is clearly seen, even for nonzero Mach numbers, when $\cos(\theta)$ approaches a value such that $M_{eff.}$ is zero.

In comparing our results with Whitham's (1974) results, we note that for spherically symmetric disturbances (for which n=0) our equations (7.8) reduce to his results. However, in more general problems (for which n>0), our results differ from Whitham's equations. In the far field, where the log term in the expression for $\tilde{F}_n$ dominates the remaining terms, equation (7.8b) reduces to

$$\tau = t-r+1 + ((\gamma+1)/2)f^{(n+1)}(\tau)\log(r_1/M)MP_n^m(\cos(\theta))\cos(m\psi-\beta).$$

For n=0, this equation is equivalent to Whitham's equation for τ. For n>0, we use $r_1/M = r$ to note that this equation is the same as Whitham's result when we replace M in his formula by $M_{eff.}$.

For n>0, the definition of τ involves terms, through the definition of $\tilde{F}_n$, which are negligibly small in the far field. However, these terms appear to have an interesting (and vital) interpretation. In particular, the terms other than the log term in $\tilde{F}_n$ give rise to terms in $D_1\tau$ which are $O(1/r^2)$ (at least) as $r\to\infty$, which, in turn, give rise to terms on the right sides of equations (7.1a,b) which are $O(1/r^3)$, as $r\to\infty$. *However, without these* *terms present, no steady state, periodic (particular) solution exists for* $\rho^{(2)}$, $u_r^{(1)}$, $u_\theta^{(1)}$, and $u_\psi^{(1)}$. To see this, we note first that, if a periodic solution with period, say, T, for these quantities does exist, we can integrate equations (7.1b)-(7.1d) over a complete period in t and, with a little manipulation, obtain the relations

$$0 = (\partial/\partial\theta)\int_0^T \Big(\text{right side of (7.1b)}\Big)dt - D_0\int_0^T \Big(\text{right side of (7.1c)}\Big)dt,$$

$$0 = (\partial/\partial\psi)\int_0^T \Big(\text{right side of (7.1b)}\Big)dt - D_0\int_0^T \Big(\text{right side of (7.1d)}\Big)dt.$$

The first term on the right side of each of these relations involves $D_1\rho^{(1)}$, while the second terms in each relation is independent of $D_1\rho^{(1)}$. For example, in the case of the oscillating sphere, the first of these relations yields the requirement

$$0 = (\tilde{a}_{1,2} + 1)(\gamma+1)\omega^4/(8r^3(4+\omega^4)).$$

Thus, if a periodic solution is to exist, we must have $\tilde{a}_{1,2} = -1 = a_{1,2}$, as stated in equation (7.8b). Similar equations hold for $n\geq 2$, which led to the particular definition of $\tilde{F}_n$ which appears in equation (7.8b).

It is also interesting to note how the distortion of the acoustical wave from a linear wave varies with the parameter n. In Figure 6 we have plotted $r\rho^{(1)} \approx r(\rho-1)/M$ at M=0.3 for (a) the pulsating sphere (n=0), (b) the oscillating sphere at $\theta=0$ (n=1), and (c) the squishing sphere at $\theta=\pi/2$ and $\psi=\pi/4$ (n=2). (In (b) and (c) the angular variables were chosen so that the distortion of the wave was maximum.) As the figure illustrates, for a given value of M the amount of distortion *decreases* as n *increases*. Intuitively, we may think of this as being due to the increasing number of "degrees of freedom" of the wave. Alternatively, to obtain the same amount of distortion in the wave as n increases, the value of M would also have to be increased. This observation is consistent with our one-dimensional example (which can be thought of as a very "confining" flow), since for this example there was significant distortion of the wave at a much lower value of the Mach number.

We also note that the analysis presented here may provide some insight into the derivation of appropriate non-reflecting boundary conditions to be used at artificial computational boundaries for a purely numerical simulation of acoustical waves generated by general (arbitrary) sources. For example, the rather simple form of our final results allows us to express the far field behavior of the waves in a concise form. This expression, when used with some of the ideas of Bayliss and Turkel (1980), for example, may allow us to derive the required boundary conditions. Investigations along these lines are continuing.

References

Bayliss, A., and Turkel, E., 1980. "Radiation boundary conditions for wave-like equations", Comm. Pure Appl. Math. 33, pp. 707-726.

Crow, S. C., 1970. "Aerodynamic Sound Emission as a Singular Perturbation Problem", Studies in Appl. Math. 49, pp. 21-44.

Lighthill, M. J., 1949. "A technique for rendering approximate solutions to physical problems uniformly valid," Phil. Mag. 40, pp. 1179-1201.

Lighthill, M. J., 1952, "On sound generated aerodynamically, I. General theory", Proc. Roy. Soc. A, 211, pp. 564-587.

Lighthill, M. J., 1954, "On sound generated aerodynamically, II. Turbulence as a source of sound", Proc. Roy. Soc. A, 222, pp. 1-32.

Lighthill, M. J., 1961. "A technique for rendering approximate solutions to physical problems uniformly valid", Z. Flugwiss. 9, pp. 267-275.

Morton, B.R., 1984, "The generation and decay of vorticity", Geophys. Astrophys. Fluid Dynamics, 28, 277-308.

Nayfeh, A. H., 1973, *Perturbation Methods*, Wiley, New York.

Ribner, H. S., 1962. "Aerodynamic sound from fluid dilations", Univ. Toronto Inst. Aerophysics Report No. 86.

Whitham, G. B., 1956. "On the propagation of weak shock waves," J. Fluid Mech. 1, pp. 290-318.

Whitham, G. B., 1974. *Linear and nonlinear waves*, Wiley (New York).

Wolfram, S., 1991, *Mathematica: A System for Doing Mathematics by Computer*, Addison-Wesley (New York).

Appendix

In this appendix we record the form of the perturbation coefficient functions for the velocity components, as well as various constants, associated with the examples presented in section 8.

Pulsating sphere:

$$u_r^{(0)} = \left((r-1)\omega/((1+\omega^2)r^2)\right)\cos(\omega\tau) + (1/(1+\omega^2))\left(\omega^2/r + 1/r^2\right)\sin(\omega\tau),$$

$$\begin{aligned} u_r^{(1)} = {} & (b_2/r^2 + 2b_3\omega/r)\cos(2\omega\tau) + (b_3/r^2 - 2b_2\omega/r)\sin(2\omega\tau) \\ & + \left((29-3\gamma)\omega^2/(16r^3(1+\omega^2)^2) + \omega^2(\omega^2-1)/(r^2(1+\omega^2)^2)\right)\cos(2\omega\tau) \\ & - ((1+\gamma)\omega^2\log(r))/(4r^2(1+\omega^2)) \\ & + \left((29-3\gamma)\omega(\omega^2-1)/(32r^3(1+\omega^2)^2) - 2\omega^3/(r^2(1+\omega^2)^2)\right)\sin(2\omega\tau) \\ & + O(1/r^4). \end{aligned}$$

Oscillating sphere:

$$\begin{aligned} u_r^{(0)} = {} & \left(2(r-1)\omega(2+r\omega^2)\cos(\omega\tau)/r^3 + (4-2\omega^2+4r\omega^2-2r^2\omega^2+r^2\omega^4)\sin(\omega\tau)/r^3\right) \\ & \bullet\cos(\theta)/(4+\omega^4), \end{aligned}$$

$$u_\theta^{(0)} = \left((\omega(2r-2-r\omega^2)\cos(\omega\tau)/r^3 + (2-\omega^2+2r\omega^2)\sin(\omega\tau)/r^3\right)\sin(\theta)/(4+\omega^4),$$

$$\begin{aligned} u_r^{(1)} = {} & R_0(r)+R_1(r)\cos(2\omega\tau)+R_2(r)\sin(2\omega\tau) + \\ & \left(S_0(r)+S_1(r)\cos(2\omega\tau)+S_2(r)\sin(2\omega\tau)\right)\cos(2\theta), \end{aligned}$$

$$u_\theta^{(1)} = \left(T_0(r)+T_1(r)\cos(2\omega\tau)+T_2(r)\sin(2\omega\tau)\right)\sin(2\theta),$$

$$f_{0c}= \omega^2\Big(2120+72\gamma+(1106-430\gamma)\omega^2+(519+7\gamma)\omega^4+(408+24\gamma)\omega^6\Big)/(384(1+4\omega^2)),$$

$$f_{0s}= -\omega\Big(2028-20\gamma+(344-680\gamma)\omega^2-(1037-499\gamma)\omega^4$$
$$+(360+104\gamma)\omega^6-512\omega^8\Big)/(768(1+4\omega^2)),$$

$$f_{2c}= \omega^2\Big(-2106+198\gamma-(141-563\gamma)\omega^2+(407-553\gamma)\omega^4-(280+104\gamma)\omega^6$$
$$+(408+24\gamma)\omega^8\Big)/(24(81+36\omega^2-32\omega^4+64\omega^6)),$$

$$f_{2s}= \omega\Big(4140-468\gamma-(6136+248\gamma)\omega^2-(4261-3291\gamma)\omega^4+$$
$$(96-480\gamma)\omega^6-(2512+336\gamma)\omega^8+512\omega^{10}\Big)/(96(81+36\omega^2-32\omega^4+64\omega^6)).$$

Here each R_j, S_j and T_j can be expressed as a multiple of (log(r))/r and a power series in the variable (1/r), beginning with a term proportional to 1/r.

Squishing sphere:

$$u_r^{(0)}= \sin(2\psi)\sin^2(\theta)\Big(((4\omega^5-9\omega^3)/r\ +(36\omega^3-4\omega^5)/r^2$$
$$+(81\omega-36\omega^3)/r^3-(81\omega-9\omega^3)/r^4)\cos(\omega t)\ +$$
$$((81-36\omega^2)/r^4+(81\omega^2-9\omega^4)/r^3-(36\omega^2-16\omega^4)/r^2-(9\omega^4-\omega^6)/r)\sin(\omega\tau))\Big)$$
$$/(81+9\omega^2-2\omega^4+\omega^6),$$

$$u_\theta^{(0)}= 2\sin(2\psi)\sin(\theta)\cos(\theta)\Big($$
$$((27\omega-3\omega^3)/r^4-(27\omega-12\omega^3)/r^3-(9\omega^3-\omega^5)/r^2)\cos(\omega\tau)\ +$$
$$((-27+12\omega^2)/r^4+(9\omega^2-4\omega^4)/r^2-(27\omega^2-3\omega^4)/r^3)\sin(\omega\tau)\Big)/(81+9\omega^2-2\omega^4+\omega^6),$$

$$u_\psi^{(0)}= 2\cos(2\psi)\sin(\theta)\Big($$

$$((27\omega-3\omega^3)/r^4-(27\omega-12\omega^3)/r^3-(9w^3-\omega^5)/r^2)\cos(\omega\tau)+$$
$$((-27+12\omega^2)/r^4+(9\omega^2-4w^4)/r^2-(27\omega^2-3\omega^4)/r^3)\sin(\omega\tau)\Big)/(81+9\omega^2-2\omega^4+\omega^6),$$

$$\rho^{(2)}=\sum_{k=0}^{2}\cos(2k\theta)\left\{\sum_{j=0}^{1}\cos(4j\psi)\left(R_{k,j}+RC_{k,j}\cos(2\omega\tau)+RS_{k,j}\sin(2\omega\tau)\right)\right\},$$

$$u_r^{(1)}=\sum_{k=0}^{2}\cos(2k\theta)\left\{\sum_{j=0}^{1}\cos(4j\psi)\left(S_{k,j}+SC_{k,j}\cos(2\omega\tau)+SS_{k,j}\sin(2\omega\tau)\right)\right\},$$

$$u_\theta^{(1)}=\sum_{k=1}^{2}\sin(2k\theta)\left\{\sum_{j=0}^{1}\cos(4j\psi)\left(T_{k,j}+TC_{k,j}\cos(2\omega\tau)+TS_{k,j}\sin(2\omega\tau)\right)\right\},$$

$$u_\psi^{(1)}=\sum_{k=1}^{2}\sin((2k-1)\theta)\Big\{\sin(4\psi)\left(U_{k,1}+UC_{k,1}\cos(2\omega\tau)+US_{k,1}\sin(2\omega\tau)\right)$$
$$+\;U_{k,0}+UC_{k,0}\cos(2\omega\tau)+US_{k,0}\sin(2\omega\tau)\Big\}.$$

Here each of the coefficients $R_{k,j}$, $RC_{k,j}$, ... , $US_{k,j}$ can be expressed as a multiple of (log(r))/r and a power series in the variable (1/r), beginning with a term proportional to 1/r.

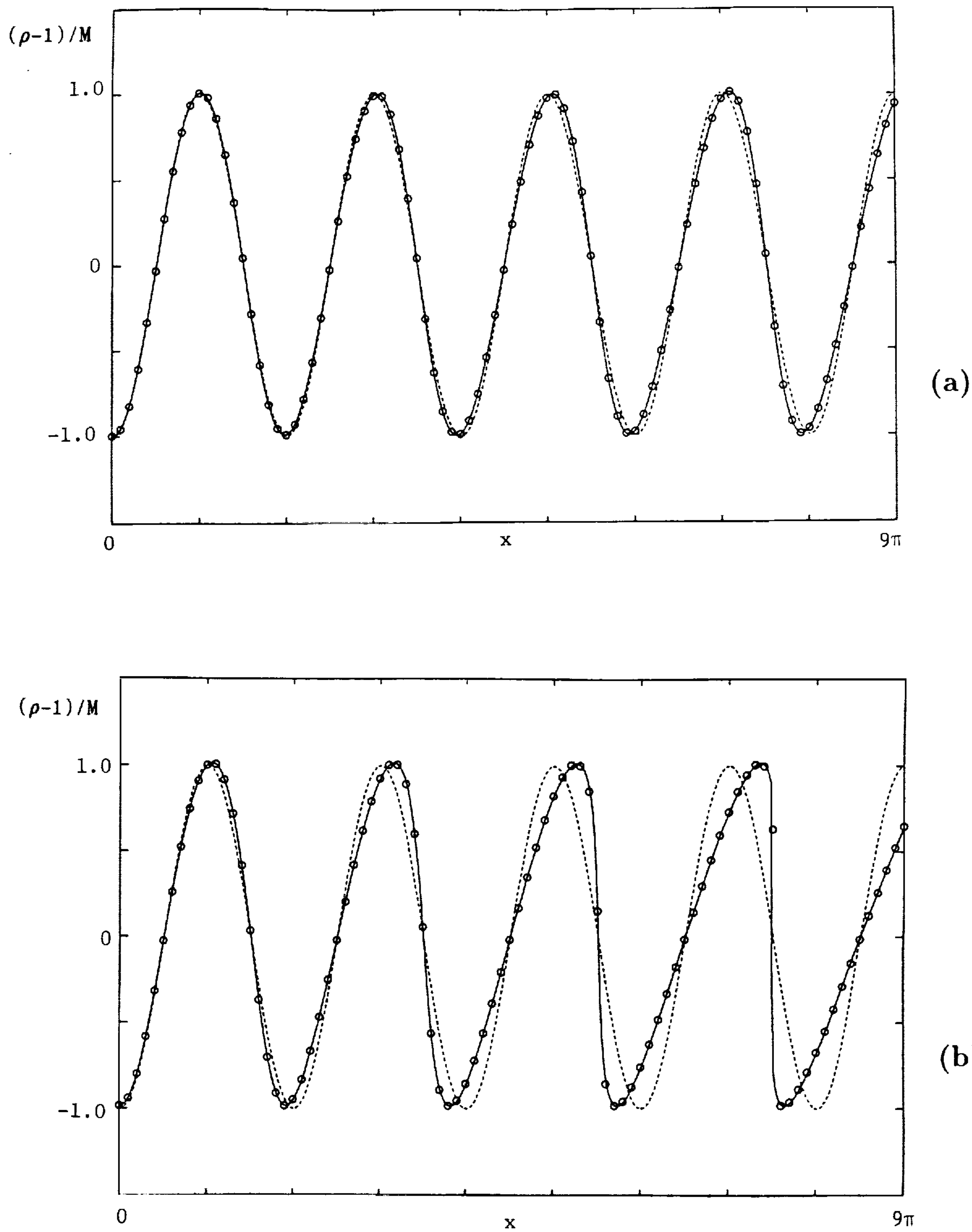

Figure 1: Approximations to $(\rho-1)/M$ for the simple one-dimensional example plotted as a function of distance from the source (at $x=0$) for $t=42.44$, using the multiple scales approximation from equation (4.18) (solid lines), the classical linear solution (4.15) (short dashed lines), and a solution to equations (4.1)-(4.2) obtained by purely numerical means (circles), with (a) $M=0.01$ and (b) $M=0.04$.

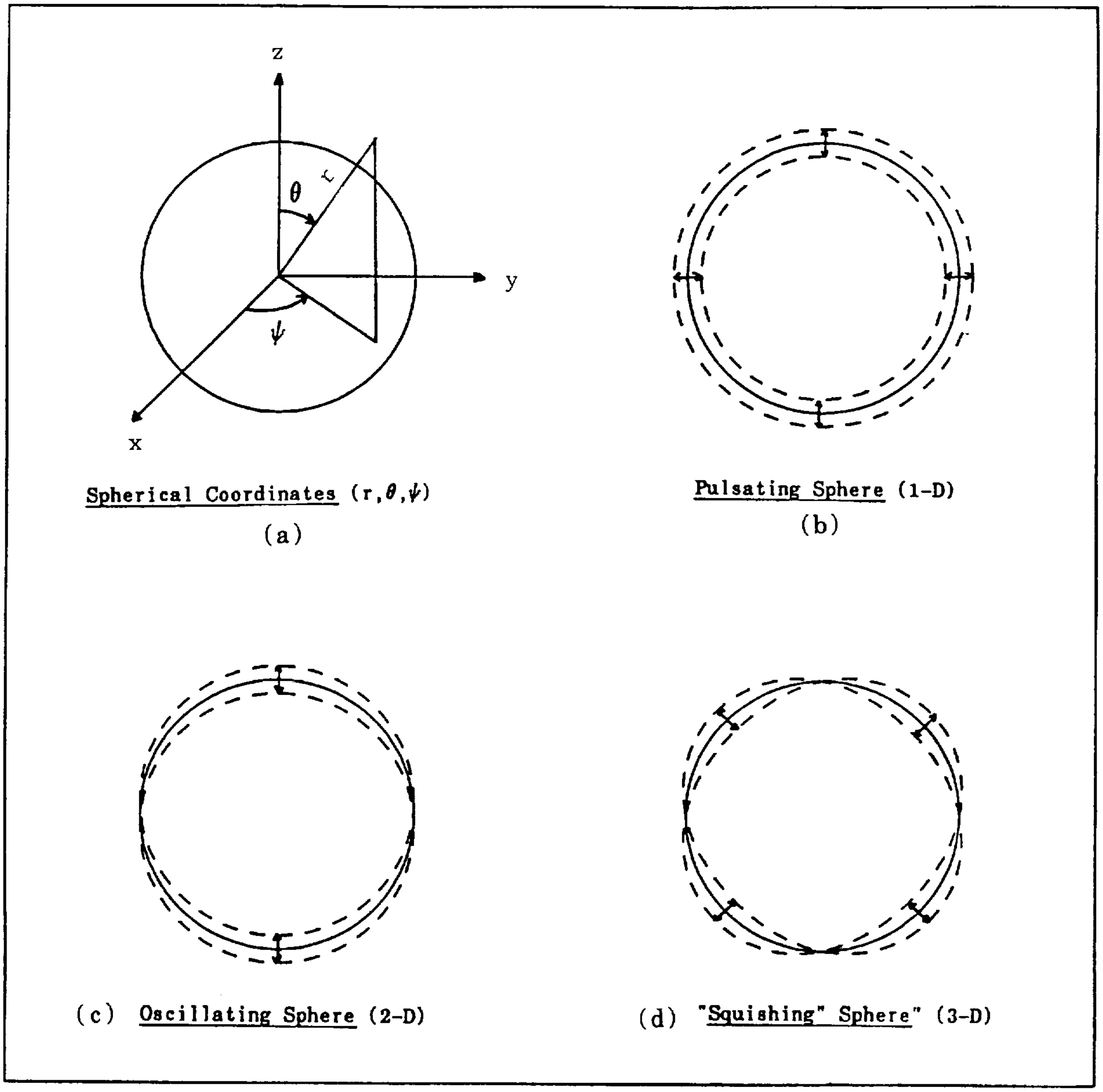

Figure 2: (a) An illustration of the Cartesian and spherical coordinate systems used in sections 5-8. Also, an indication of the vibrations of the sphere for the three examples considered in section 8: (b) the pulsating sphere; (c) the oscillating sphere; and (d) the "squishing" sphere, indicating the motion of the sphere in the plane $\theta=\pi/2$.

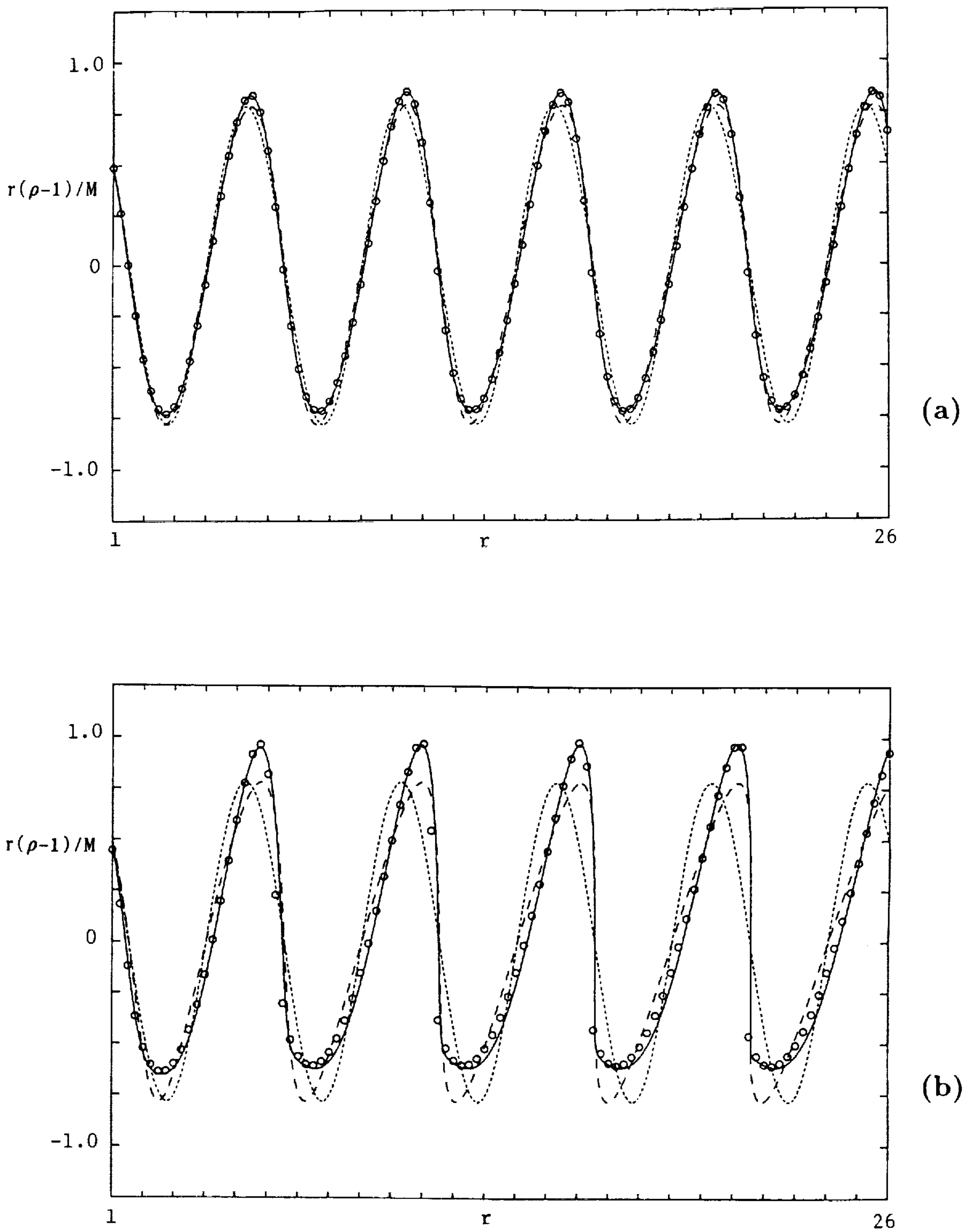

Figure 3: Approximations to $r(\rho-1)/M$ for the pulsating sphere plotted as a function of distance from the center of the sphere at t=50 and for $\omega=1.25$ using the multiple scales solution $r(\rho^{(1)}+M\rho^{(2)})$ (solid lines), the classical linear solution (short dashed lines), the approximation based on Whitham's (1974) first order solution (long dashed lines), and a solution obtained by purely numerical methods (circles), for (a) M=0.1 and (b) M=0.3.

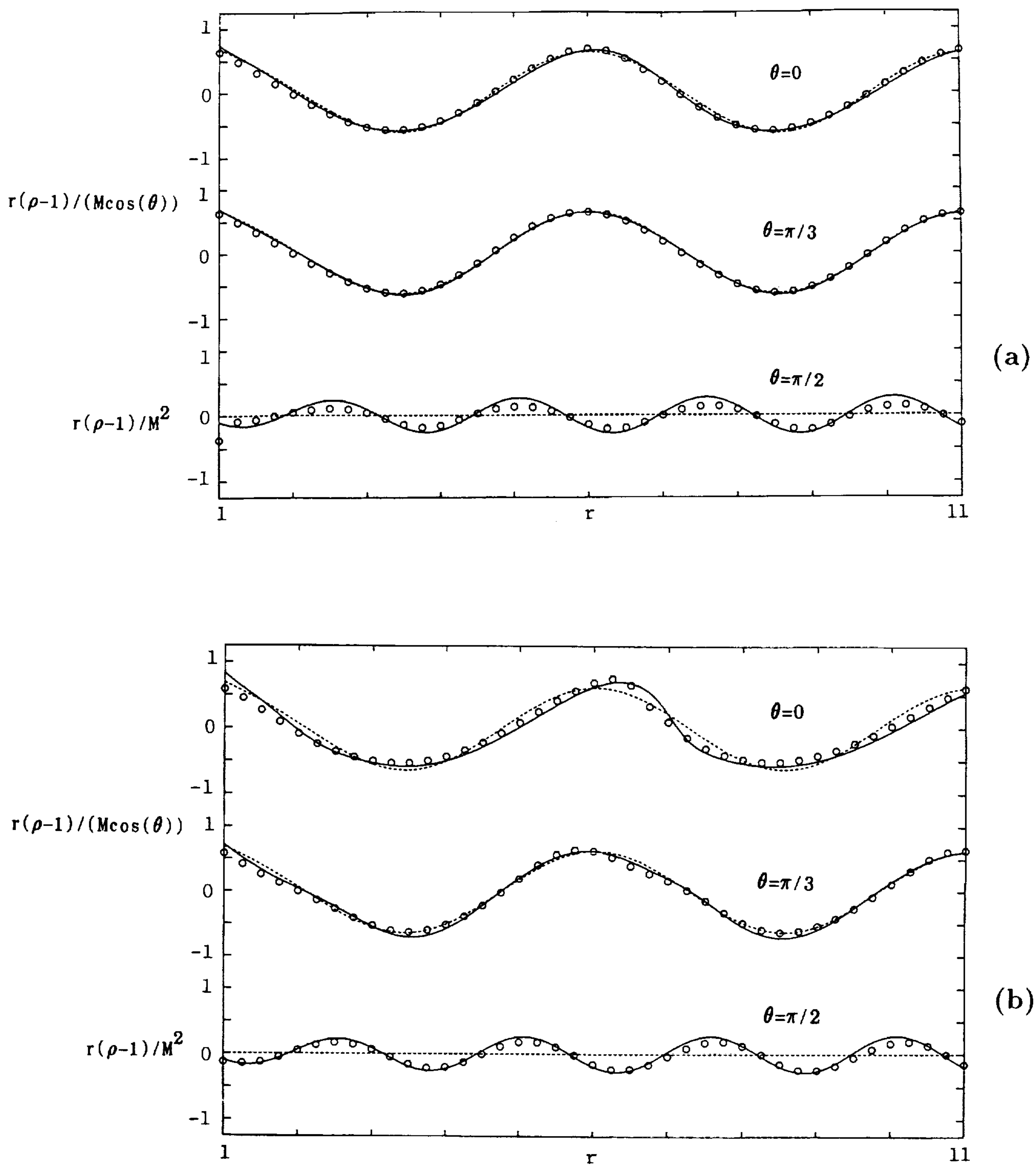

Figure 4: Approximations to $r(\rho-1)/(M\cos(\theta))$ for the oscillating sphere for $\theta=0$ and $\theta=\pi/3$, as well as approximations to $r(\rho-1)/M^2$ at $\theta=\pi/2$ plotted as a function of distance from the center of the sphere at $t=25$ and for $\omega=1.25$ using the multiple scales solution (solid lines), the classical linear solution (short dashed lines), and a solution obtained by purely numerical means (circles), for (a) $M=0.1$ and (b) $M=0.3$.

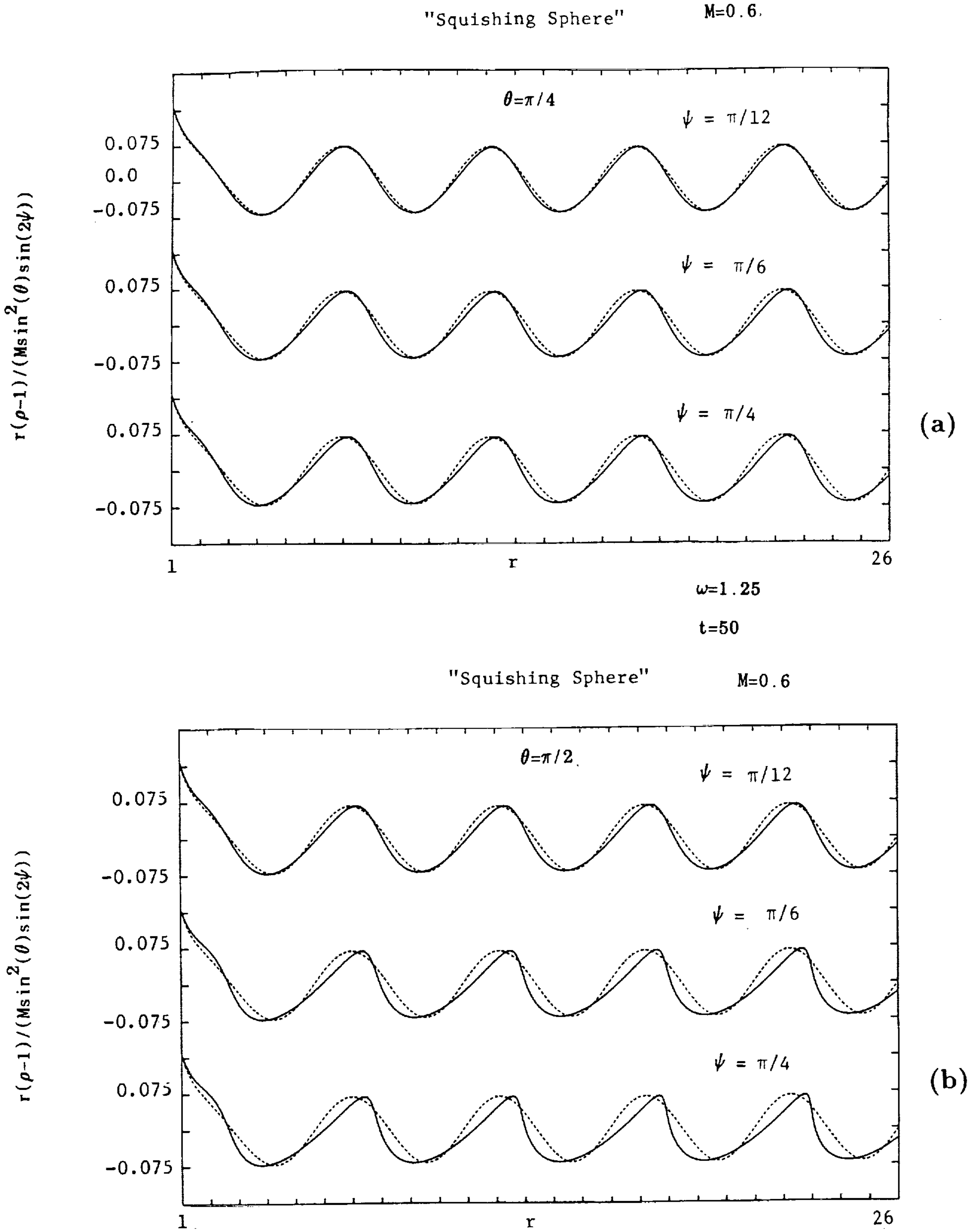

Figure 5: Approximations to $r(\rho-1)/(M\sin^2(\theta)\sin(2\psi))$ for the "squishing sphere" example plotted as a function of distance from the center of the sphere at $t=50$ with $\omega=1.25$ and $M=0.6$, for (a) $\theta=\pi/4$ and (b) $\theta=\pi/2$, at $\psi = \pi/12$, $\pi/6$, and $\pi/4$, using the multiple scales solution $r\rho^{(1)}/(3\sin^2(\theta)\sin(2\psi)$ (solid lines) and the classical linear solution (dashed lines).

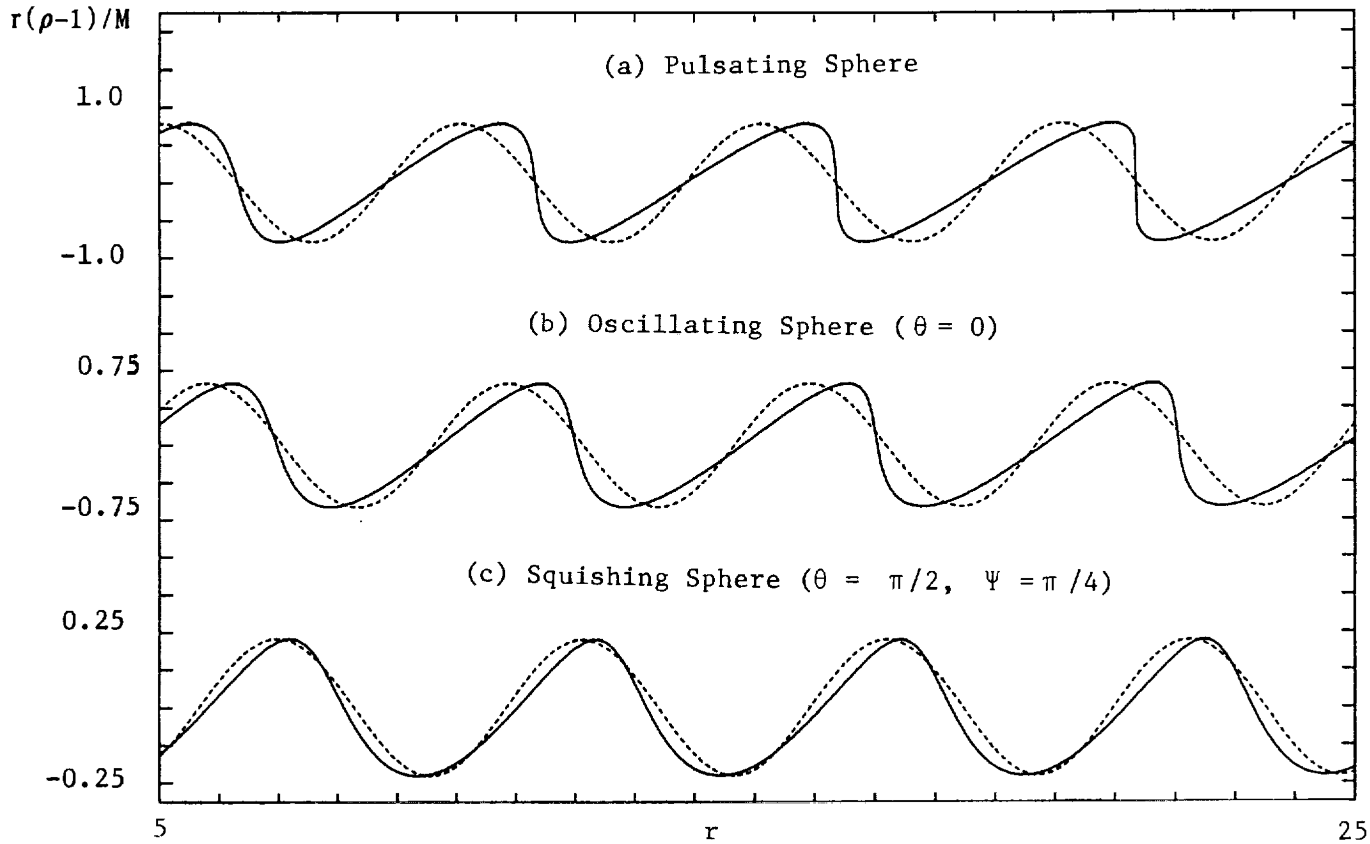

Figure 6: Comparison of $r\rho^{(1)} \approx r(\rho-1)/M$ with M=0.3, t=50, and ω=1.25 for: (a) the pulsating sphere, (b) the oscillating sphere at θ=0; and (c) the squishing sphere at $\theta=\pi/2$ and $\psi=\pi/4$. In each case, the corresponding linear solution is plotted as a dashed line.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
December 1992

3. REPORT TYPE AND DATES COVERED
Contractor Report

4. TITLE AND SUBTITLE
A MULTIPLE SCALES APPROACH TO SOUND GENERATION BY VIBRATING BODIES

5. FUNDING NUMBERS
C NAS1-18605
C NAS1-19480

WU 505-90-52-01

6. AUTHOR(S)
James F. Geer
Dennis S. Pope

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Institute for Computer Applications in Science and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23681-0001

8. PERFORMING ORGANIZATION REPORT NUMBER
ICASE Report No. 92-73

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23681-0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
NASA CR-191414
ICASE Report No. 92-73

11. SUPPLEMENTARY NOTES
Langley Technical Monitor: Michael F. Card
Final Report

To be submitted to Journal of Fluid Mechanics

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified - Unlimited

Subject Category 34

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

The problem of determining the acoustic field in an inviscid, isentropic fluid generated by a solid body whose surface executes prescribed vibrations is formulated and solved as a multiple scales perturbation problem, using the Mach number M based on the maximum surface velocity as the perturbation parameter. Following the idea of multiple scales, new "slow" spacial scales are introduced, which are defined as the usual physical spacial scale multiplied by powers of M. The governing nonlinear differential equations lead to a sequence of linear problems for the perturbation coefficient functions. However, it is shown that the higher order perturbation functions obtained in this manner will dominate the lower order solutions unless their dependence on the slow spacial scales is chosen in a certain manner. In particular, it is shown that the perturbation functions must satisfy an equation similar to Burgers' equation, with a slow spacial scale playing the role of the time-like variable. The method is illustrated by a simple one-dimensional example, as well as by three different cases of a vibrating sphere. The results are compared with solutions obtained by purely numerical methods and some insights provided by the perturbation approach are discussed.

14. SUBJECT TERMS
aeroacoustics; vibrating sphere; multiple scales

15. NUMBER OF PAGES
45

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified		

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std Z39-18
298-102

NASA-Langley, 1993